

Practical and theoretical improvements for bipartite matching using the pseudoflow algorithm

Bala G. Chandran

Analytics Operations Engineering, Inc.

Boston, MA 02109.

bchandran@nltx.com

Dorit S. Hochbaum

Department of Industrial Engineering and Operations Research and

Walter A. Haas School of Business

University of California

Berkeley, CA 94720.

hochbaum@ieor.berkeley.edu

Abstract

We show that the pseudoflow algorithm for maximum flow is particularly efficient for the bipartite matching problem both in theory and in practice. We develop several implementations of the pseudoflow algorithm for bipartite matching, and compare them over a wide set of benchmark instances to state-of-the-art implementations of push-relabel and augmenting path algorithms that are specifically designed to solve these problems. The experiments show that the pseudoflow variants are in most cases faster than the other algorithms.

We also show that one particular implementation—the matching pseudoflow algorithm—is theoretically efficient. For a graph with n nodes, m arcs, n_1 the size of the smaller set in the bipartition, and the maximum matching value $\kappa \leq n_1$, the algorithm’s complexity given input in the form of adjacency lists is $O(\min\{n_1\kappa, m\} + \sqrt{\kappa} \min\{\kappa^2, m\})$. Similar algorithmic ideas are shown to work for an adaptation of Hopcroft and Karp’s bipartite matching algorithm with the same complexity. Using boolean operations on words of size λ , the complexity of the pseudoflow algorithm is further improved to $O(\min\{n_1\kappa, \frac{n_1 n_2}{\lambda}, m\} + \kappa^2 + \frac{\kappa^{2.5}}{\lambda})$. This run time is faster than for previous algorithms such as Cheriyan and Mehlhorn’s algorithm of complexity $O(\frac{n^{2.5}}{\lambda})$.

1 Introduction

The bipartite matching problem is to find, in a given bipartite graph $B = (V_1; V_2, E)$, a matching containing a maximum number of edges. That is, a collection of edges $M \subseteq E$ such that each node is adjacent to at most one of the edges in the matching M . For a survey on early literature on this problem the reader is referred to the book by Lawler [23], Chapter 5.

The bipartite matching problem is equivalent to the maximum flow problem on an associated *simple* bipartite network. (A network is said to be simple if every node has a throughput capacity of 1 unit of flow.) Therefore, any maximum flow algorithm can be used to solve the bipartite matching problem. The network is constructed by adding source and sink nodes s and t , linking the source to all nodes of V_1 with arcs of capacity 1 and all nodes of V_2 to the sink with arcs of capacity 1, and directing all edges in the bipartite graph from V_1 to V_2 with capacity ≥ 1 . Such a network is shown in Figure 1. The maximum

s, t -flow on this associated network corresponds to a solution to the maximum matching problem: an edge $[i, j] : i \in V_1, j \in V_2$ is in the matching if and only if the corresponding arc (i, j) has a flow of one unit on it.

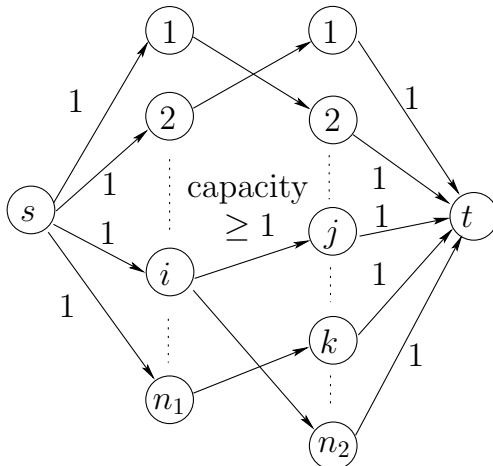


Figure 1: Flow graph for bipartite matching.

Other than bipartite matching there are other well-known problems that are solved as maximum flow on simple bipartite networks. These include the vertex cover problem on a bipartite graph and the independent set problem, also known as the stable set problem, on a bipartite graph. We refer to the maximum flow algorithm for simple bipartite graphs as the bipartite matching algorithm although it applies to these problems as well.

Dinic's [14] maximum flow algorithm is particularly efficient for simple networks as demonstrated by Even and Tarjan [15]. For bipartite graphs, the running time is $O(\sqrt{n_1}m)$, where $n_1 = |V_1|$ (w.l.o.g. $n_1 \leq n_2 = |V_2|$). Hopcroft and Karp [21] proposed an algorithm for bipartite matching with complexity $O(\sqrt{\kappa}m)$, where κ is the cardinality of the maximum matching which is bounded by n_1 . Their algorithm is, in essence, the same as Dinic's algorithm adapted to bipartite matching. Feder and Motwani [16] obtained a bound of $O(\sqrt{nm^*})$ for the bipartite matching algorithm that relies on speeding up Dinic's algorithm using graph compression. In the complexity expression, m^* is the number of edges in the compressed graph, which is less than m by about a factor of $\log n$. Using boolean word operations on λ -bit words, Cheriyan and Melhorn [11] obtained a bound of $O(\frac{n^{2.5}}{\lambda})$ while Alt et al. [5] obtained a bound of $O(n^{1.5}\sqrt{\frac{m}{\lambda}})$ (which is better than $O(\frac{n^{2.5}}{\lambda})$ for sparse graphs and better than $O(\sqrt{nm})$ for dense graphs). Mucha and Sankowski [24] described a randomized algorithm for matching in general (non-bipartite) graphs that runs in $O(n^\omega)$, where $\omega = 2.38$ is the exponent of the best known matrix multiplication algorithm.

However, the theoretically efficient algorithms listed above tend to perform poorly in practice. Setubal [25, 26] showed that in practice, implementations of the push-relabel algorithm of Goldberg and Tarjan [18] were faster than those of Dinic's as well as the algorithm of Alt et al. Cherkassky et al. [12] developed several implementations of push-relabel and performed extensive experiments on several benchmark instances, showing push-relabel to be the fastest in practice.

In this paper, we apply the pseudoflow algorithm of Hochbaum [19, 20] to bipartite matching and examine its theoretical and practical performance. The pseudoflow algorithm was recently shown by Chandran and Hochbaum [9] to be the fastest algorithm in practice for the maximum flow problem, and by Hochbaum and Orlin to be as efficient as the push-relabel algorithm in theory; hence, it is reasonable to suspect that the pseudoflow algorithm is efficient for bipartite matching as well. The major contributions of our work are as follows.

1. We develop several implementations of the pseudoflow algorithm specifically for bipartite matching and show that are faster than state-of-the-art implementations of push-relabel for bipartite matching. We use the results of the experiments to gain insights into the differences between the pseudoflow and push-relabel algorithms.

2. We show that a variant of the pseudoflow algorithm, called the **matching-pseudoflow** algorithm, runs on a bipartite simple network in time $O(\min\{n_1\kappa, m\} + \sqrt{\kappa} \min\{\kappa^2, m\})$. We then show that the insights generated from this approach allow to modify either Hopcroft and Karp’s algorithm or the push-relabel maximum flow algorithm and achieve the same complexity. Using boolean operations on λ -bit words, we show that the complexity of the **matching-pseudoflow** algorithm can be further improved to $O\left(\min\left\{m, n_1\kappa, \frac{n_1n_2}{\lambda}\right\} + \kappa^2 + \frac{\kappa^{2.5}}{\lambda}\right)$.

Since the **matching-pseudoflow** algorithm could be viewed as a superior implementation of Dinic’s algorithm, we compare the performance of the **matching-pseudoflow** to the best-known implementation of Dinic’s algorithm to understand and quantify the key differences between the two algorithms.

2 Description of the pseudoflow algorithm

The pseudoflow algorithm and its properties are described in detail in Hochbaum [20]. The description is repeated here for completeness.

2.1 Preliminaries

Let G_{st} be a graph $(V \cup \{s, t\}, A \cup A_s \cup A_t)$, where A_s and A_t are the source-adjacent and sink-adjacent arcs respectively.

A flow vector $f = \{f_{ij}\}_{(i,j) \in A \cup A_s \cup A_t}$ is said to be *feasible* if it satisfies

1. Flow balance constraints: for each $i \in V$, $\sum_{(k,i) \in A \cup A_s \cup A_t} f_{ki} = \sum_{(i,j) \in A \cup A_s \cup A_t} f_{ij}$ (i.e., $\text{inflow}(i) = \text{outflow}(i)$), and
2. Capacity constraints: the flow value is between the lower bound and upper bound capacity of the arc, i.e., $\ell_{ij} \leq f_{ij} \leq u_{ij}$. Without loss of generality, we assume henceforth that $\ell_{ij} = 0$ (e.g., Ahuja et al. [3], pages 191–196).

A *maximum flow* is a feasible flow f^* that maximizes the flow out of the source (or into the sink). The value of the maximum flow is $\sum_{(s,i) \in A_s} f_{si}^*$.

Given a flow vector f in G_{st} that is feasible, the *residual graph* $G^f = (V \cup \{s, t\}, A^f)$ is constructed as follows: for each arc $(i, j) \in A \cup A_s \cup A_t$ with flow f_{ij} and capacity c_{ij} , A^f contains two arcs: (i, j) with capacity $c_{ij} - f_{ij}$ and (j, i) with capacity f_{ij} . The capacities of arcs in A^f are referred to as the residual capacities with respect to flow f , and are denoted by c^f . An s, t -cut in the graph is a bi-partition of nodes into two disjoint sets – one containing the source and the other containing the sink. One property of the residual graph is that the bipartition of nodes of the minimum s, t -cut of G^f is the same as that in G (e.g., Ahuja et al. [3], pages 44–46).

A *pseudoflow* f is a flow vector that satisfies capacity constraints, but may violate flow balance at any node. The *excess* of a node $v \in V$ is the inflow into that node minus the outflow denoted by $e(v) = \sum_{(u,v) \in A \cup A_s \cup A_t} f_{uv} - \sum_{(v,w) \in A \cup A_s \cup A_t} f_{vw}$. A negative excess is called a *deficit*.

A tree $T = (V, E)$ is a connected, undirected, acyclic graph. A rooted tree has a distinguished node w called the root. For each edge $[u, v]$, u is said to be the *parent* of v if u is closer to the root than v , and is denoted by $\text{parent}(v)$. Node v is then called the *child* of u , and is denoted by $\text{child}(u)$. The only node in the tree that does not have a parent is the root. A node v is said to be an *ancestor* of a node u if v lies along the unique path from v to the root; node u is then said to be a *descendant* of node v . For convenience, we will assume that the tree points topologically “downward” with the root at the “top” of the tree, and each node “below” its ancestors. A *branch* rooted at some node r is a sub-graph of the tree that contains r and all its descendants in the tree. A rooted *sub-tree* is a connected sub-graph of the given tree (unlike a branch, it need not contain all the descendants of its root).

An arc that carries a flow equal to its upper bound is said to be *saturated*. The pseudoflow algorithm maintains a flow that saturates source-adjacent and sink-adjacent arcs throughout the algorithm. Consequently, the source and sink have no further role in the algorithm and are contracted into a single node r that “keeps track” of the excesses and deficits of the nodes in V by adding excess and deficit arcs as follows:

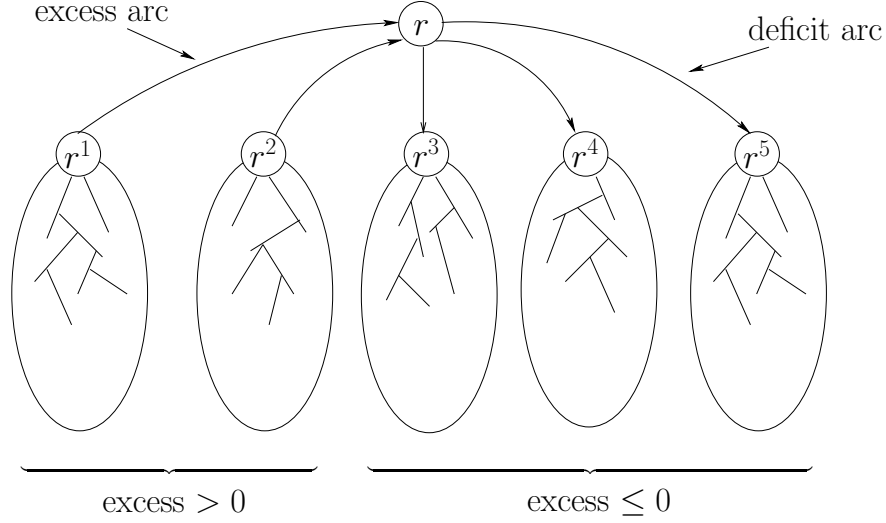


Figure 2: A schematic description of a normalized tree. Each r^i is the root of a branch.

For each node $v \in V$ with positive excess, we add to the graph an arc (v, r) called an *excess arc*, and for each node $u \in V$ with negative excess we add an arc (r, u) called a *deficit arc*. The network thus obtained is referred to as the extended network $G^{ext} = (V \cup \{r\}, A \cup A_r)$, where A_r is the set of excess and deficit arcs.

For a tree T , an arc (u, v) is said to be *in-tree* if the edge $[u, v] \in T$. Arcs that are not in tree are said to be *out-of-tree*. Given a pseudoflow f that saturates A_s and A_t , a *normalized tree* is a tree in G^{ext} rooted at r that satisfies the following three properties.

Property 2.1 *The nodes that do not satisfy flow balance constraints are the children of r and are the roots of their respective branches.*

Property 2.2 *The pseudoflow values of f on out-of-tree arcs are at the lower or upper bound capacities of the respective arcs.*

Property 2.3 *In every branch, all downward residual capacities are strictly positive.*

A schematic description of a normalized tree is shown in Figure 2.

The pseudoflow algorithm starts with any normalized tree and an associated pseudoflow. The generic initialization is the *simple* initialization: source-adjacent and sink-adjacent arcs are saturated while all other arcs have zero flow.

If a node v is both source-adjacent and sink-adjacent, then at least one of the arcs (s, v) or (v, t) can be pre-processed out of the graph by sending a flow of $\min\{c_{sv}, c_{vt}\}$ along the path $s \rightarrow v \rightarrow t$. This flow eliminates at least one of the arcs (s, v) and (v, t) in the residual graph. We henceforth assume w.l.o.g. that no node is both source-adjacent and sink-adjacent.

The simple initialization creates a set of source-adjacent nodes with excess, and a set of sink-adjacent nodes with deficit. Since all other arcs have zero flow, they are all out-of-tree arcs. Thus, each node is a singleton branch for which it serves as the root, even if it is *balanced* (with 0-deficit). The simple initialization results in a simple normalized tree shown in Figure 3.

2.2 A labeling pseudoflow algorithm

In the labeling pseudoflow algorithm, all nodes carry a label ℓ_v for all $v \in V$. Initially, all labels are set to the value 1. An iteration of the algorithm consists of identifying a branch with root carrying strictly positive excess, and attempting to push this excess towards the sink through the residual network. The process of pushing excesses towards the sink is performed via a *merger*. Given a branch with root of label ℓ and positive

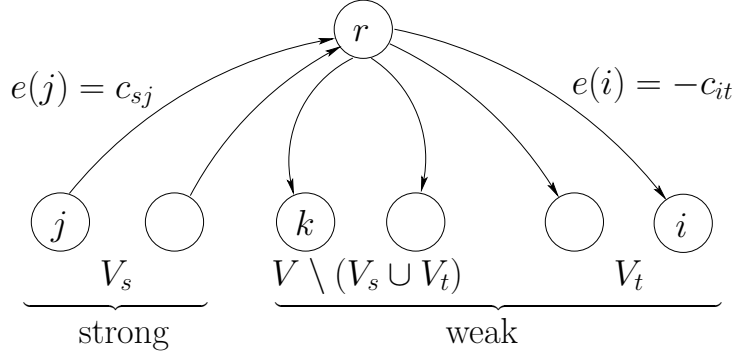


Figure 3: A simple normalized tree.

excess, a merger operation consists of identifying a *merger arc* with positive residual capacity from a node of label ℓ within the branch to some node of label $\ell - 1$ in the graph.

A *relabeling* of a node is the increase of a node's label by one unit. A node of label ℓ is relabeled to $\ell + 1$ if there is no merger arc in the residual graph to a neighbor of label $\ell - 1$, and if all its children in the branch have label at least $\ell + 1$. With these rules, the labels satisfy the following properties.

Lemma 2.1 (Hochbaum [19, 20]) *For the labeling pseudoflow algorithm, the labels satisfy:*

(a) *For every residual arc (u, v) , $\ell_u \leq \ell_v + 1$.*

(b) *The labels of nodes are monotone nondecreasing in the downwards direction in each branch.*

Corollary 2.1 (Hochbaum [19, 20]) *The label assigned to a node throughout the labeling pseudoflow algorithm does not exceed the length of a shortest path to a sink-adjacent node in the residual graph plus the label of the sink-adjacent node. More generally, the positive difference in labels of two nodes does not exceed the length of the residual path between them.*

For convenience, we henceforth refer to the branch containing the tail of the merger arc as the “from-branch” and the one containing the head of the merger arc as the “to-branch”. Once a merger arc is identified, a merger operation is performed on the normalized tree. This consists of adding the merger arc to the normalized tree, and removing the arc from the root of the from-branch to the root of the normalized tree. The merger operation is shown in Figures 4(a) and (b). At the end of a merger, the tree is not a normalized tree since it has a non-root node carrying positive excess. The merged branch is now *renormalized*, a process that may create any number of branches out of the merged branch. The process of renormalization of the merged branch consists of pushing the excess of the root of the from-branch towards the root of the to-branch and updating the pseudoflows and excesses. The path from the root of the from-branch to that of the to-branch is unique since they are nodes in a connected tree. For each edge on this path, the operation of pushing the excess from the child to its parent and updating the pseudoflow on the edge is called a *push*. If only a part of the child's excess can be pushed to its parent due to insufficient residual capacity on that arc, the child retains some positive excess. The edge to its parent is then removed from the normalized tree and an excess arc is added for the child node making it the root (with positive excess) of a branch consisting of all nodes below it. This operation, called a *split*, is shown in Figures 4(b) and 4(c).

If the root of a branch is relabeled to label n at some point in the algorithm, all nodes in this branch have label n . By Corollary 2.1, this implies that all deficit nodes are unreachable from nodes of label n . Hence, all nodes in the branch must be in the source set of a minimum cut, and can be ignored for the remainder of the algorithm. Thus, the algorithm terminates when (i) there are no branches with root carrying positive excess, or (ii) all such roots have a label of n .

When the algorithm terminates, we obtain a normalized tree and a pseudoflow where all nodes belonging to branches with positive excess (if they exist) have label n . This is not a feasible flow since the normalized tree has excess and deficits. However, the normalized tree contains information regarding a minimum cut, which is stated in the following theorem.

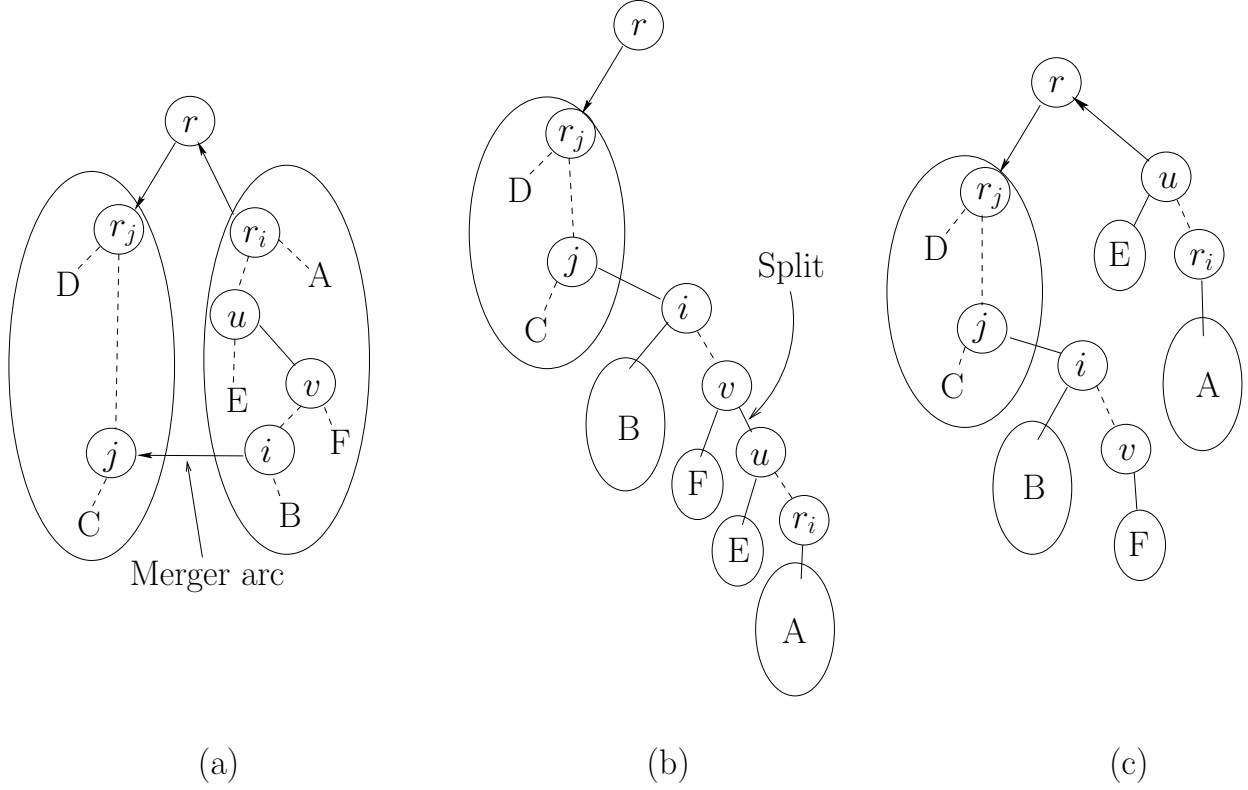


Figure 4: (a) Initial normalized tree, (b) Tree obtained after the merger, (c) Re-normalized tree after split due to insufficient residual capacity on edge $[v, u]$.

Theorem 2.1 (Hochbaum [19, 20]) *The source node along with all nodes of label n in the normalized tree form the source set of a minimum cut while the remaining nodes form the sink set of a minimum cut.*

2.3 Implementation details

Limiting the number of arc scans: During the labeling algorithm, the arcs adjacent to each node are examined at most once (see Hochbaum [19, 20]) for each value of the node's label. To implement this, we maintain a pointer at each node to the arc that was last scanned to find a merger. If any node is visited more than once for a given label, the search for mergers resumes from the last scanned arc, thus ensuring that each arc is scanned at most once for each label. When a node is relabeled, the pointer is reset to the start of its list of adjacent arcs.

Root management: The labeling algorithm requires that all roots with positive excess and of a particular label be available when queried. To achieve this, the roots are maintained in an array of buckets, where a bucket contains all roots with positive excess and with a particular label. The order in which roots within a bucket are processed for mergers appears to make a difference to the pseudoflow algorithm. Anderson and Hochbaum [6] experimented with three branch management policies:

- **FIFO:** Each bucket is maintained as a queue; roots are added to the rear of the queue, and roots are retrieved from the front of the queue.
- **FIFO:** Each bucket is maintained as a stack; roots are added to the top of the stack, and roots are retrieved from the top of the stack.
- **Wave:** This is a variant of the LIFO policy. Each bucket is still maintained as a stack, with roots being added to the top of the stack and being retrieved from the top. However, when the excess of a

root changes while it is in the bucket, it is moved up to the top of the stack.

Note that the wave management policy is the same as the LIFO policy for the lowest label variant since the excess of a root with positive excess does not change while it is in a bucket. (When a root is processed in the lowest label algorithm, all mergers are from a branch with positive excess to one with non-positive excess, leaving all other roots with positive excess unchanged.)

Gap Relabeling: We use the gap-relabeling heuristic of Derigs and Meier [13], who introduced it in the context of push-relabel. When we process a branch whose root has label ℓ and there are no nodes in the graph with label $\ell - 1$, we conclude that the entire branch has no residual paths to the sink and is hence a part of the source set of a min cut. The entire branch can thus be ignored for the rest of the algorithm. In practice, this is achieved by setting the labels of all nodes in that branch to n .

The *Min-cut Stage* refers to all the operations executed until a minimum cut is obtained.

2.4 Lowest and highest label pseudoflow variants

In the generic labeling algorithm, the branch with a root carrying positive excess that is selected for processing (finding mergers) is chosen arbitrarily. In the *lowest label variant*, the root carrying positive excess with the lowest label is identified and the branch is processed for mergers so long as its root remains the lowest labeled root with positive excess. In the *highest label variant*, the branch that is chosen is the one with root of highest label, i.e., at each iteration the root carrying positive excess with highest label is identified and that branch is processed. Note that in the lowest label variant, the root of the from-branch has positive excess while that of the to-branch has non-positive excess, while in the highest label variant, roots of both the from-branch and to-branch could have positive excess.

3 Complexity of the pseudoflow algorithm for bipartite matching

We now analyze the complexity of the highest and lowest label pseudoflow algorithms when applied to bipartite matching.

Definition 3.1 *The algorithm is said to be in phase ℓ when nodes of label ℓ are being examined for mergers.*

Let the cardinality of the maximum matching in G be κ . Since the graph is bipartite, every alternate node in any path in the network must be a V_2 -node. The shortest path from any node in the network to a node with strict deficit (i.e., an unmatched V_2 node) can contain at most κ matched V_2 -nodes, hence its length is at most 2κ . By Corollary 2.1 this means that the label of each node (and hence the number of phases) for the lowest label algorithm is $O(\kappa)$, while that for the highest label algorithm is $O(n_1)$.

Proposition 3.1 *The depth of the normalized tree is $O(\kappa)$.*

Proof: Consider a path from a node up to the root of the normalized tree. Since the graph is bipartite, the path is made up of alternating nodes from V_1 and V_2 . Thus, every alternate edge in the path is a valid matching, which bounds the length of the path (and thus the depth of the tree) by 2κ . ■

The implication of the above proposition is that the work done per merger is $O(\kappa)$.

Proposition 3.2 *The number of arc scans in the lowest pseudoflow algorithm for bipartite matching is $O(\min\{\kappa m, n_1^2 \kappa\})$.*

Proof: Hochbaum [19, 20] showed that each arc is examined $O(1)$ times per phase. Since there are $O(\kappa)$ phases and m arcs, the total number of arc scans is $O(\kappa m)$.

Each time a node is processed, its neighbors are examined in order to find a merger. Since there are $O(n_1)$ nodes in the normalized tree, at most n_1 neighbors need to be examined in order to find a merger or determine that no merger exists. Thus, the total number of arc scans is the number of nodes in the normalized tree times the number of arc scans per phase times the number of phases, which is $O(n_1^2 \kappa)$. ■

Similarly, for the highest label algorithm, the number of arc scans $O(\min\{n_1m, n_1^3\})$.

Following the pseudopolynomial complexity analysis of the generic lowest label pseudoflow algorithm from Hochbaum [19, 20], we get a bound of $O(n_1\kappa)$ on the number of mergers for the lowest label variant. The number of mergers in the highest label pseudoflow algorithm is $O(n_1m)$ (as shown by Hochbaum [19, 20], the number of mergers is bounded by m times the number of phases).

The total work done in the pseudoflow algorithm is the number of arc scans plus the number of mergers times work per merger (which is $O(\kappa)$ as shown above). Thus, the complexity of the lowest label pseudoflow algorithm for bipartite matching is $O(\min\{\kappa m, n_1^2\kappa\} + n_1\kappa^2)$, while that of the highest label algorithm is $O(\kappa n_1m)$.

4 The free-arcs pseudoflow algorithm for bipartite matching

In the *free-arcs* version of the pseudoflow algorithm, the normalized tree satisfies the following property in addition to Properties 2.1 through 2.3.

Property 4.1 *In every branch, all upward residual capacities are strictly positive.*

The only difference from the perviously described pseudoflow algorithm is in the **split** operation, which is now initiated if the upward residual capacity of an in-tree arc becomes zero after a push.

The implication of the above property is that the normalized tree contains only “free” arcs, i.e., arcs that have flow strictly between their lower and upped bounds.

Given a bipartite graph $G = (V_1; V_2, E)$, the flow network is constructed by adding source and sink nodes s and t , linking the source to all nodes of V_1 with arcs of capacity 1 and all nodes of V_2 to the sink with arcs of capacity 1, and directing all edges in the bipartite graph from V_1 to V_2 with *infinite* capacity. For the free-arcs algorithm, the infinite capacity on arcs from V_1 to V_2 implies that all in-tree arcs have unit flow while all out-of-tree arcs have flow equal to the lower bound of zero (the flow on an arc can never be at its upper bound).

Lemma 4.1 *The pseudoflow algorithm for bipartite matching can create only four types of branches – two types of strong branches ST_1 and ST_2 , and two types of weak branches WT_1 and WT_2 (as described in Figure 5(a)).*

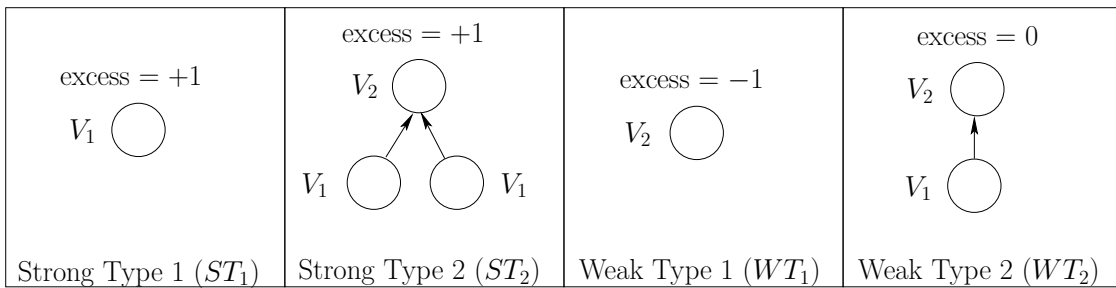


Figure 5: Types of branches that can exist in a normalized tree during execution of the free-arcs pseudoflow algorithm for bipartite matching.

Proof: The proof is by induction. The inductive assumption applies initially as in the simple normalized tree all nodes of V_1 are ST_1 branches and all nodes of V_2 are WT_1 branches. Given that an iteration starts with these two types of strong branches and two types of weak branches, only four types of mergers are possible as shown in Figure 5(b). All these mergers result in one or two of these types of branches, and thus the proof is complete. ■

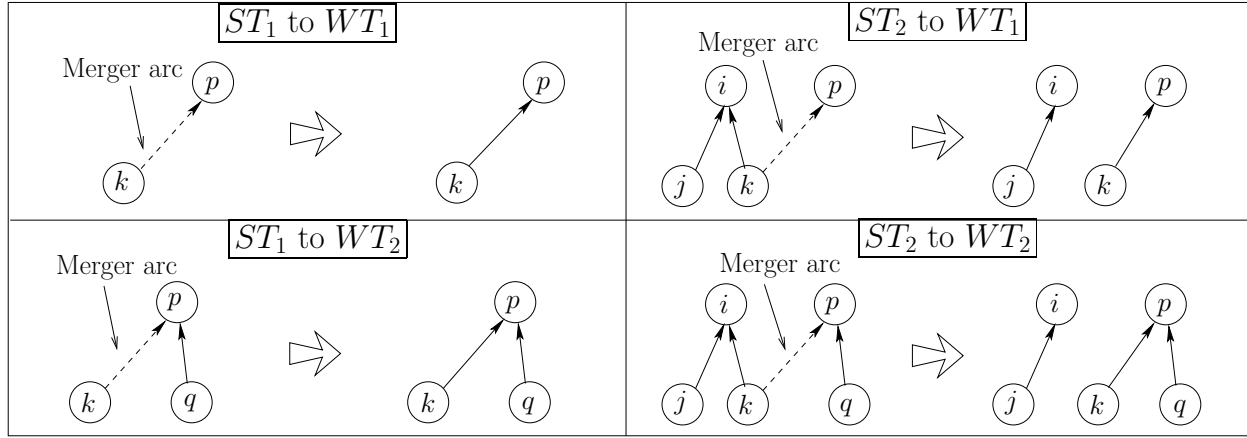


Figure 6: Types of mergers that can occur in the free-arcs pseudoflow algorithm for bipartite matching.

Each WT_2 and ST_2 branch contains an edge between a V_1 node and a V_2 node, and all branches are node-disjoint. Thus, the set of WT_2 and ST_2 branches represent a valid matching, which leads to the following property.

Property 4.2 *The number of ST_2 and WT_2 branches is bounded by κ , the cardinality of the maximum matching.*

4.1 Complexity of the free-arcs pseudoflow algorithm for bipartite matching

All the results in Section 3 are still valid, except that the work done per merger is now $O(1)$. The complexity of the free-arcs version of the pseudoflow algorithm is thus the number of arc scans plus the number of mergers, which is $O(\min\{\kappa m, n_1^2 \kappa\})$ for the lowest label variant and $O(n_1 m)$ for the highest label variant.

5 The matching-pseudoflow algorithm

The matching-pseudoflow algorithm is a pseudoflow algorithm with global relabeling and delayed relabeling. We first introduce the concept of two-edge distance labels in the graph.

Definition 5.1 *The two-edge distance label of a node is the number of V_1 -nodes in the shortest path from that node to a WT_1 branch in the residual network.*

The notion of two-edge distances in bipartite graphs has been used previously e.g. by Ahuja et al. [4]. Initially, all labels of nodes in V_1 , which are ST_1 branches, are set to 1 and the labels of nodes in V_2 , which are WT_1 branches, are set to 0. Throughout the algorithm the labels of nodes that form WT_1 branches remain 0 as their two-edge distance (to themselves) is 0.

Delayed relabeling means that all possible mergers from lowest labeled strong nodes of label ℓ are performed *without relabeling the nodes* when no merger is found. Once all the nodes of label ℓ have been examined for mergers, all the node labels are set to be the shortest two-edge distance to a WT_1 branch in the residual graph and the set of all mergers starting from the lowest labeled strong root are again performed. The process of computing all the node distance labels is referred to as *global relabeling* [17].

We now demonstrate that the two-edge distance labels satisfy properties analogous to (a) and (b) of Lemma 2.1. Property (a) is satisfied by the distance labels of nodes which are the lengths of the shortest residual path from each node to the sink. The second property of monotonicity (b) is shown to be satisfied next.

Lemma 5.1 *The two-edge distance labels satisfy property (b) in Lemma 2.1.*

Proof: Two-edge distance labels satisfy that *both nodes in a WT_2 branch have the same label*: In a WT_2 branch, all arcs into its root (a V_2 -node) other than that from its child (a V_1 -node) carry zero flow; the arc from its child carries a flow of 1 unit. The arc from its child is the only arc with positive residual capacity adjacent to the root. Thus, the root can reach a WT_1 node only through its child and the shortest path from the root to a WT_1 branch will contain the shortest path from its child to a WT_1 branch. So the number of V_1 -nodes in the shortest path from the root to a WT_1 branch will be the same as that in the shortest path from its child to a WT_1 branch, ensuring that the root and child have the same two-edge distance label.

A similar argument holds for the ST_2 branches. Let ℓ_R be the label of the right child and ℓ_L be the label of the left child (assume w.l.o.g. that $\ell_R \leq \ell_L$). The root of an ST_2 branch can reach a WT_1 branch only through one of its children; so the label of the root from a WT_1 branch will be equal to ℓ_R , the smaller label of the two children. Since there is a residual arc (of infinite capacity) from the left child to the root, the two-edge distance from the left child to the right is 1. Hence, $\ell_L \leq \ell_R + 1$, and the label of the left child is at least equal to and at most one greater than the label of the root. ■

Definition 5.2 Stage ℓ of the algorithm is the maximal set of mergers that occur while the shortest two-edge distance from a strong node to a WT_1 branch is ℓ .

An initialization procedure, equivalent to a stage 1, creates a maximal set of WT_2 branches by scanning the neighbors of each V_1 -node to identify an unmatched V_2 -node and then performing a merger. Since the cardinality of the maximum matching is at most κ , we need to scan at most κ neighbors of each V_1 -node to identify an unmatched V_2 -node or determine that none exists. Also, each arc is scanned at most once, so the complexity of this procedure is $O(\min\{m, n_1\kappa\})$. At the end of the initialization, the shortest two-edge distance from a ST_1 branch to a WT_1 branch is at least 2.

We now elaborate on the implementation of a stage. To facilitate the description, we introduce the following notation for labels of nodes in a branch. The labels of an ST_2 branch are represented by the triplet $(left, root, right)$ which represent the labels of the left child, root, and right child respectively. We will assume w.l.o.g. that the left child has label greater than or equal to that of the right child. Labels in a WT_2 branch are represented by the pair $(child, parent)$ which represent the labels of the child and parent respectively.

At the beginning of each stage, global relabeling is performed, and all nodes are “unflagged”, which marks them as being *unvisited*. Mergers are allowed only between unvisited nodes.

The merger/split operations at each stage are such that they satisfy the property that the stage begins and ends with only WT_1 , ST_1 , and WT_2 branches; ST_2 branches are only formed temporarily during a stage. This inductive property holds initially for stage 2 since no ST_2 branches are formed in stage 1 (the greedy initialization).

Suppose that at the beginning of stage $\ell \geq 2$, the set of branches consists of ST_1 branches of label $\geq \ell$; WT_2 branches in which both the nodes have the same label p ($1 \leq p < \ell$); and WT_1 branches which have label 0. Consider a sequence of mergers starting from a ST_1 branch of lowest label ℓ . The first merger is from a ST_1 branch of label ℓ to an unvisited root of a WT_2 branch with label $(\ell - 1, \ell - 1)$. This creates a ST_2 branch $(\ell, \ell - 1, \ell - 1)$. This branch now has the lowest labeled strong root, and the search for mergers starts from the right child labeled $\ell - 1$.

Suppose that at some point a merger results in a ST_2 branch $(p + 1, p, p)$. The search for mergers now starts from the right child of this branch resulting in one of the following possible outcomes.

1. There is no merger to an unvisited weak node of label $p - 1$: Here we *delay* the relabeling of that node to the end of the stage and mark the root of the branch as being *visited* implying that the branch cannot participate in any more mergers at the current stage. In this case, a *backtrack* operation is performed to reverse the last merger and restore the structure of the branches to what it was prior to the last merger. This is shown in Figure 7. For example, consider the case where the backtrack operation occurs from a branch of label $(\ell, \ell - 1, \ell - 1)$ in stage ℓ . Suppose there are no mergers from the right child of this branch, then the backtrack operation splits the branch, creating a WT_2 branch of label $(\ell - 1, \ell - 1)$ and one ST_1 branch of label ℓ . The root of the WT_2 branch is marked as visited, and the search for mergers continues from the ST_1 branch. If no more mergers are possible from this node, it is marked as visited, and a new lowest labeled strong node is picked. This procedure continues until there are no more unvisited ST_1 nodes of label ℓ .

2. $p > 1$ and a merger is found to an unvisited root of a WT_2 branch of label $(p-1, p-1)$: This creates a ST_2 branch of label $(p, p-1, p-1)$ and the search for mergers continues from the right child of this branch.
3. $p = 1$ and a merger is found to a WT_1 branch which has label 0: This creates a new WT_2 branch $(1, 0)$, incrementing the size of the current matching. The branches involved in this sequence of mergers are all marked as visited, and do not participate in any more mergers in stage ℓ . The process of searching for mergers then starts with an unvisited ℓ labeled strong node if there is one, or else the stage terminates.

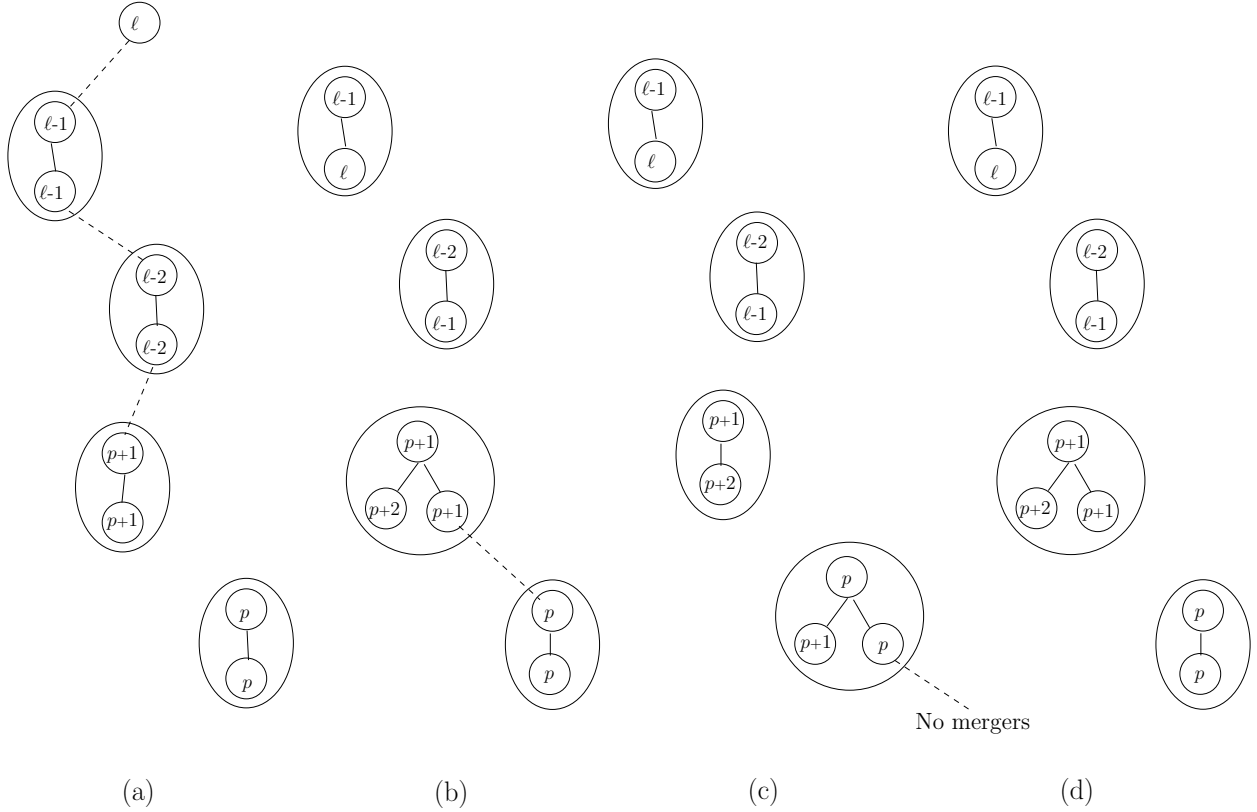


Figure 7: (a) Branches before mergers (shown in dotted lines), (b) Branches after mergers until label $p+1$, (c) Branches after merger from $p+1$ to p , (d) Branches after lack of merger causes a backtrack. The branch with label p is marked as having been visited.

We have thus proved the following lemma, which holds inductively given that stage 1 ends with ST_1 , WT_1 , and WT_2 branches.

Lemma 5.2 *With the merger/split/backtrack operations described above, a stage that begins with only WT_1 , WT_2 , and ST_1 branches terminates with only these three branch types.*

Definition 5.3 *A successful path of length ℓ is a sequence of mergers, at stage ℓ , that starts at a ST_1 branch of label ℓ and ends at a WT_1 branch of label 0.*

A successful path contributes to the increase of the number of WT_2 branches by 1 which is equivalent to increasing the size of the matching. The mergers that form a successful path are called *successful mergers*. The next lemma proves that the procedure of flagging nodes as visited does not block off any successful paths, implying that *all* successful paths of length ℓ are found in stage ℓ .

Lemma 5.3 *A node that is marked as visited in stage ℓ can no longer be part of a successful path of length ℓ .*

Proof: If a V_2 -node has been marked as visited after no merger have been found from its right child, then that child cannot lead to any merger in the current stage. Hence, once a backtrack occurs, the V_2 -node is marked as visited, and need not be visited again during that stage.

If an unvisited V_2 -node, v of label p belongs to a successful path then it has a child of label $p + 1$ after the successful merger. If v were to participate in another successful merger at the same stage, the sequence of labels in this second successful path would be $\ell \rightarrow (\ell - 1) \rightarrow \dots \rightarrow (p + 1) \rightarrow p \rightarrow (p + 1) \rightarrow p \rightarrow \dots \rightarrow 0$. The length of such a path is strictly greater than a two-edge distance ℓ since layer p is visited twice in this path. Thus, each V_2 -node participates in at most one successful path of length ℓ in stage ℓ . Hence, a node that was part of a successful merger is marked as visited, and need not be visited again during that stage. ■

Corollary 5.1 *Each V_2 -node participates in at most one successful merger or one backtrack at each stage.*

Corollary 5.2 *The labels of all lowest labeled strong nodes of label ℓ at stage ℓ strictly increase after the termination of stage ℓ .*

Performing global relabeling is equivalent to generating a so-called *layered network* (as in Dinic's maximum flow algorithm). In a layered network, each layer consists of all branches with a particular label. In stage ℓ , layer 0 consists of all nodes that have distance label 0, i.e., only WT_1 branches. Layers 1 through $\ell - 1$ consist of WT_2 branches, and layer ℓ consists of ST_1 branches.

We now describe the procedure for generating the layered network, which is the critical part of our algorithm. Let the k -layer ($0 \leq k \leq \ell$) be the set of nodes with label k . The layered network can be generated by scanning all backward residual arcs from the sink using a Breadth-First-Search (BFS).

In a naive implementation of BFS one would start with all WT_1 branches (label 0) and look at all incoming arcs in the residual network to generate the 1-layer. This could take $O(m)$ work and is expensive. An alternative approach is to check for each WT_2 branch whether it is in the 1-layer by checking if there is an arc from its child node to a WT_1 branch. Using the fact that labels are non-decreasing, we only need to check this for WT_2 branches that were of label 1 in the previous layered network.

Generating the 1-layer: The neighbors of each V_1 child node of label 1 in a WT_2 branch are scanned to identify a residual arc to a WT_1 branch. Since there are at most κ WT_2 branches, we need to scan at most κ neighbors of each V_1 -node of label 1 to identify a WT_1 neighbor node or determine that none exists (in which case the branch does not belong to the 1-layer). If a WT_1 node is not adjacent to a V_1 -node of label 1 in stage ℓ then it cannot be adjacent to that V_1 -node in any later stage. This is because no new WT_1 branches are created in any stage. Hence, each arc needs to be scanned at most once throughout the algorithm. By maintaining a pointer for each V_1 -node to the last arc scanned at each stage, and resuming the search from that arc in the next stage, we can ensure that each arc is scanned at most once throughout the algorithm.

Claim 5.1 *The total work done to generate the 1-layer throughout the algorithm is $O(\min\{\kappa^2, m\})$.*

Generating layers 2 through $\ell - 1$: Given the set of WT_2 branches in layer p , the incoming residual arcs into the root of each WT_2 branch in layer p are examined to obtain neighbors in layer $p + 1$. Scanning the incoming residual arcs of a WT_2 branch stops if an ST_1 neighbor is found, since then $p = \ell - 1$ and the WT_2 branch is in the $\ell - 1$ layer its ST_1 neighbor is thus in the ℓ -layer.

There are at most κ incoming arc scans for each WT_2 root required to label all the WT_2 branches in the next layer or find a ST_1 branch. Since there are at most κ roots of WT_2 branches and each arc is scanned at most once in each stage, so total work done in generating layers 2 through $\ell - 1$ at each stage is $O(\min\{\kappa^2, m\})$.

Claim 5.2 *The work done per stage to generate the layers 2 through $\ell - 1$ is $O(\min\{\kappa^2, m\})$.*

The layered network generated has all layers of weak branches up to the $\ell - 1$ -layer, and some ST_1 branches in the ℓ -layer. This ℓ -layer may not contain all the ST_1 branches of label ℓ since not all incoming

arcs to the WT_2 branches were examined. However, by Corollary 5.1, it is sufficient to have at most one neighbor ST_1 branch for every WT_2 branch of label $\ell - 1$. Therefore, instead of explicitly generating the entire ℓ -layer, once an ST_1 branch of label ℓ is found, it is determined that the WT_2 branch is in the $\ell - 1$ -layer – the last layer of weak branches.

For each unvisited WT_2 branch of label $\ell - 1$, we scan its incoming arcs to check for an ST_1 neighbor. If such an ST_1 branch is found, a sequence of mergers is initiated from this strong branch. If a successful path is found, or if no more mergers are possible from this strong branch, another unvisited WT_2 branch in the $\ell - 1$ layer is chosen and its incoming arcs are scanned to identify a new ST_1 branch from which mergers are initiated. This continues until all the WT_2 branches in the $\ell - 1$ layer have been visited or have been scanned for a neighboring ST_1 branch.

There are at most κ branches in the $\ell - 1$ layer. For each such branch, at most 2κ incoming arcs need to be scanned to identify a new neighboring ST_1 branch. Also, each arc in the network is examined at most once, so the work done per stage in identifying the necessary ST_1 branches in the ℓ -layer is $O(\min\{m, \kappa^2\})$.

Each arc participates in at most one merger per stage, each of which requires $O(1)$ work; mergers thus require $O(\min\{\kappa^2, m\})$. A backtrack operation is performed at most once for each WT_2 branch, so work done in backtracking is $O(\kappa)$ per stage.

Lemma 5.4 *The work done per stage including generating the layered network, mergers, and backtrack operations is $O(\min\{\kappa^2, m\})$.*

Lemma 5.5 *The number of stages in the algorithm is $O(\sqrt{\kappa})$.*

The proof is along the lines of those of Even and Tarjan [15] and Hopcroft and Karp [21]. Details are provided in Section A of the appendix.

Theorem 5.1 *For input given in the form of adjacency lists, the complexity of the matching-pseudoflow algorithm is $O(\min\{n_1\kappa, m\} + \min\{\kappa^2, m\}\sqrt{\kappa})$.*

Proof: The complexity of initialization is $O(\min\{n_1\kappa, m\})$. There are $O(\sqrt{\kappa})$ stages in the algorithm, each of which takes $O(\min\{\kappa^2, m\})$. The work to generate layer 1 is $O(\min\{\kappa^2, m\})$ throughout the algorithm. The total complexity is therefore $O(\min\{n_1\kappa, m\} + \min\{\kappa^2, m\}\sqrt{\kappa})$. ■

A high-level description of the matching-pseudoflow algorithm is given in Figure 8.

Note that the matching-pseudoflow algorithm could be viewed as an efficient implementation of Dinic's algorithm with two-edge pushes: a successful path of mergers is essentially an augmenting path, while the procedure for generating the layered network is the same once greedy initialization has been performed. Similarly, the matching-pseudoflow algorithm could also be interpreted as an implementation of push-relabel with two-edge pushes that uses delayed relabeling and global relabeling.

5.1 Matching-pseudoflow with word operations

The complexity of the matching-pseudoflow algorithm can be further improved to $O\left(\min\{n_1\kappa, \frac{n_1 n_2}{\lambda}, m\} + \kappa^2 + \frac{\kappa^{2.5}}{\lambda}\right)$ using boolean word operations, where λ is the length of a word, as done by Cheriyan and Mehlhorn [11]. The key idea is to represent the graph adjacency structure using words and performing boolean operations on these words to find merger arcs. Details are provided in Section B of the appendix.

5.2 A combined algorithm

We follow the approach of Alt et al. [5] to combine the matching-pseudoflow with and without words to describe new complexity bounds. The new bound is obtained by applying the matching-pseudoflow without words until a certain stage ℓ and then using word operations for the rest of the algorithm. The greedy initialization procedure is performed with words, which has a complexity of $O(\min\{n_1\kappa, \frac{n_1 n_2}{\lambda}, m\})$. The words SUB-IN and SUB-OUT described in section B are also constructed irrespective of the algorithm used. These two operations have a complexity of $O(\min\{n_1\kappa, \frac{n_1 n_2}{\lambda}, m\} + \kappa^2)$.

```

/*
The procedure finds a maximum matching in a bipartite graph  $G = (V_1; V_2, E)$ . It terminates with
a set of  $ST_1$ ,  $WT_1$ , and  $WT_2$  branches; the set of edges in the  $WT_2$  branches form a maximum
cardinality matching.
/

procedure matching-pseudoflow:
  begin
    Generate a greedy maximal matching of  $ST_1$ ,  $WT_1$ , and  $WT_2$  branches;
    Generate a layered network;
    Mark all nodes in the layered network as unvisited;
    while the lowest label of an  $ST_1$  branch is less than  $|V_1|$  do
      while  $\exists$  a lowest labeled unvisited  $V_1$ -node  $v$  of label  $\ell$  do
        if  $\exists$  a merger from  $v$  to an unvisited node of label  $(\ell-1)$  do
          Perform merger (as in Figures 7(a)–(b));
          if merger leads to an augmentation do
            Mark all nodes along the successful path as visited;
          else do
            Mark branch containing node  $v$  as visited;
            Perform backtrack (as in Figures 7(c)–(d));
          Generate a new layered network;
          Mark all nodes in the layered network as unvisited;
        end
  end

```

Figure 8: High-level description of the matching-pseudoflow algorithm.

Case 1: $\kappa^2 \in O(m)$

The work done until stage ℓ without using words is $O(\ell\kappa^2)$. Following analysis similar to that in the proof of Lemma 5.5, the remaining number of stages is κ/ℓ . The work done beyond stage ℓ using word operations is $O(\kappa^{2.5}/\lambda)$. The value of ℓ that minimizes the total work done is obtained by solving for ℓ in the equation $\ell\kappa^2 = \frac{\kappa}{\ell} \frac{\kappa^2}{\lambda}$.

This yields $\ell = \sqrt{\kappa/\lambda}$ and an overall complexity of $O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\} + \kappa^2 + \frac{\kappa^{2.5}}{\sqrt{\lambda}})$, which is dominated by the complexity of the algorithm with word operations. Thus, when $\kappa^2 \in O(m)$, combining the two algorithms does not provide any benefit.

Case 2: $\kappa^2 \in \Omega(m)$

The work done until stage ℓ is $O(\ell m)$. We again find the best value of ℓ by solving $\ell m = \frac{\kappa}{\ell} \frac{\kappa^2}{\lambda}$, which gives $\ell = \kappa\sqrt{\frac{\kappa}{m\lambda}}$. This leads to an overall complexity of $O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\} + \kappa^2 + \kappa^{1.5}\sqrt{\frac{m}{\lambda}})$.

This is better than the $\sqrt{\kappa m}$ complexity when $\kappa^2 \in O(m\lambda)$. Table 1 summarizes the complexity results.

Algorithm	Best when	Complexity
With word operations	$\kappa^2 \in O(m)$	$O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\} + \kappa^2 + \kappa^{2.5}/\lambda)$
Combined	$\kappa^2 \in \Omega(m) \cup O(\lambda m)$	$O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\} + \kappa^2 + \kappa^{1.5}\sqrt{\frac{m}{\lambda}})$
Without word operations	$\kappa^2 \in \Omega(\lambda m)$	$O(\min\{n_1\kappa, m\} + \sqrt{\kappa} \min\{\kappa^2, m\})$

Table 1: Summary of complexity results for the matching-pseudoflow algorithm.

Note that the complexity expressions for the matching-pseudoflow algorithm without words, the combined algorithm, and the matching-pseudoflow algorithm with words are correspondingly faster than the algorithms of Hopcroft and Karp [21] with complexity $O(\sqrt{\kappa m})$, Alt et al. [5] with complexity $O(n^{1.5}\sqrt{\frac{m}{\lambda}})$, and Cheriyan and Mehlhorn [11] with complexity $O(\frac{n^{2.5}}{\lambda})$.

6 An experimental study

6.1 Implementations

We developed eight pseudoflow implementations for bipartite matching:

1. Five “regular” pseudoflow implementations—highest label with FIFO buckets (`pseudo_hi_fifo`), highest label with LIFO buckets (`pseudo_hi_lifo`), highest label with Wave buckets (`pseudo_hi_wave`), lowest label with FIFO buckets (`pseudo_lo_fifo`), and lowest label with LIFO buckets (`pseudo_lo_lifo`).
2. Two “free-arcs” variants—`pseudo_hi_free` and `pseudo_lo_free` that are the highest and lowest label implementations of the free-arcs pseudoflow algorithm. Both these implementation use LIFO buckets, which were found to be fastest in initial testing. We use a global relabeling heuristic that periodically re-computes distance labels to all V_1 -nodes in the graph.
3. The `matching-pseudoflow` algorithm.

The latest version of the code (version 1.01) is available at [8].

Cherkassky et al. [12] developed the following algorithms for bipartite matching that implement “two-edge” pushes:

- `bim_dfs` and `bim_bfs`: These two variants apply a simple depth-first-search and breadth-first-search respectively to find augmenting s - t paths.
- `pr_bim_hi`, `pr_bim_lo`, and `pr_bim_fifo`: These are implementations of the highest label, lowest label, and FIFO push-relabel variants respectively.
- `bim_ar`: The “augment-relabel” algorithm could be thought of as a hybrid between an augmenting path algorithm and push-relabel. It is similar in spirit to the basic algorithm described by Alt et al. [5] for the bipartite matching problem.
- `bim_lds`: The “label-directed-search” variant uses a depth-first-search along with “approximate” distance labels that are periodically updated using global relabeling.

In addition, we tested `dinic`, an implementation of Dinic’s algorithm by Setubal [25], and `abmp`, a simplified implementation by Setubal [26] of the algorithm of Alt et al.[5] that is available as part of the BIPM solvers for bipartite matching [1]. While `dinic` was shown to have poor performance in practice, we use it mainly to compare it to `matching-pseudoflow`, which is its closest pseudoflow counterpart.

The pseudoflow codes, `dinic`, and `abmp` were written in C and compiled with the `gcc` compiler while those of Cherkassky et al. [12] were written in C++ and compiled used the `g++` compiler. The `-O4` compiler optimization flag was used in all cases.

6.2 Computing environment

The experiments were run on a Sun UltraSPARC workstation with a 270 MHz CPU and 192 MB of RAM. The results of the machine calibration experiment as suggested by the First Dimacs Implementation Challenge [2] are shown in Table 2.

	Test 1			Test 2		
	real	user	system	real	user	system
No optimization	0.4	0.4	0.0	3.3	3.3	0.0
-O4 flag	0.2	0.1	0.0	2.0	1.9	0.0

Table 2: Average running times for Dimacs machine calibration tests.

6.3 Differences between matching-pseudoflow and dinic in practice

As noted earlier, the **matching-pseudoflow** algorithm has parallels to Dinic's algorithm (global relabeling is equivalent to generating a layered network in Dinic's algorithm, and a successful path is equivalent to an s - t augmenting path). However, what sets the **matching-pseudoflow** algorithm apart from Dinic's algorithm is the manner in which global relabeling is performed. In this section, we demonstrate that the global relabeling procedure which leads to a better theoretical complexity also makes a significant difference in practice.

In the **matching-pseudoflow**, only the nodes in the current matching and their adjacent edges are examined during global relabeling, whereas in Dinic's algorithm the entire network (including all unmatched V_2 -nodes) and their adjacent arcs are examined to construct the layered network. Therefore, in practice, we would expect the run-time of the **matching-pseudoflow** algorithm to be dependent largely on κ , while that of Dinic's algorithm to be dependent on m .

We implemented the **matching-pseudoflow** algorithm and compared it to the best-known implementation of Dinic's algorithm for bipartite matching [25]. Instances were generated in the following manner: given n_1 , n_2 , and the expected number of edges \bar{m} , a graph is generated where each of the possible $n_1 n_2$ edges exists independently with probability $\bar{m}/(n_1 n_2)$. We generated problems with $n_1 = 16384$, and n_2/n_1 ranging from 1 to 1.2 in steps of 0.01. Thus, the most unbalanced graph had 19661 nodes. For each of the 20 classes, we generated graphs with expected number of edges 81920, 163840, 245760, and 327680 respectively, which resulted in κ being exactly or very close to n_1 . For each of the combinations of n_2 and m , we generated 10 instances and the time for each instance was averaged over 5 runs. Thus, each data point is the average of 50 runs. The run-times are shown in Figure 9.

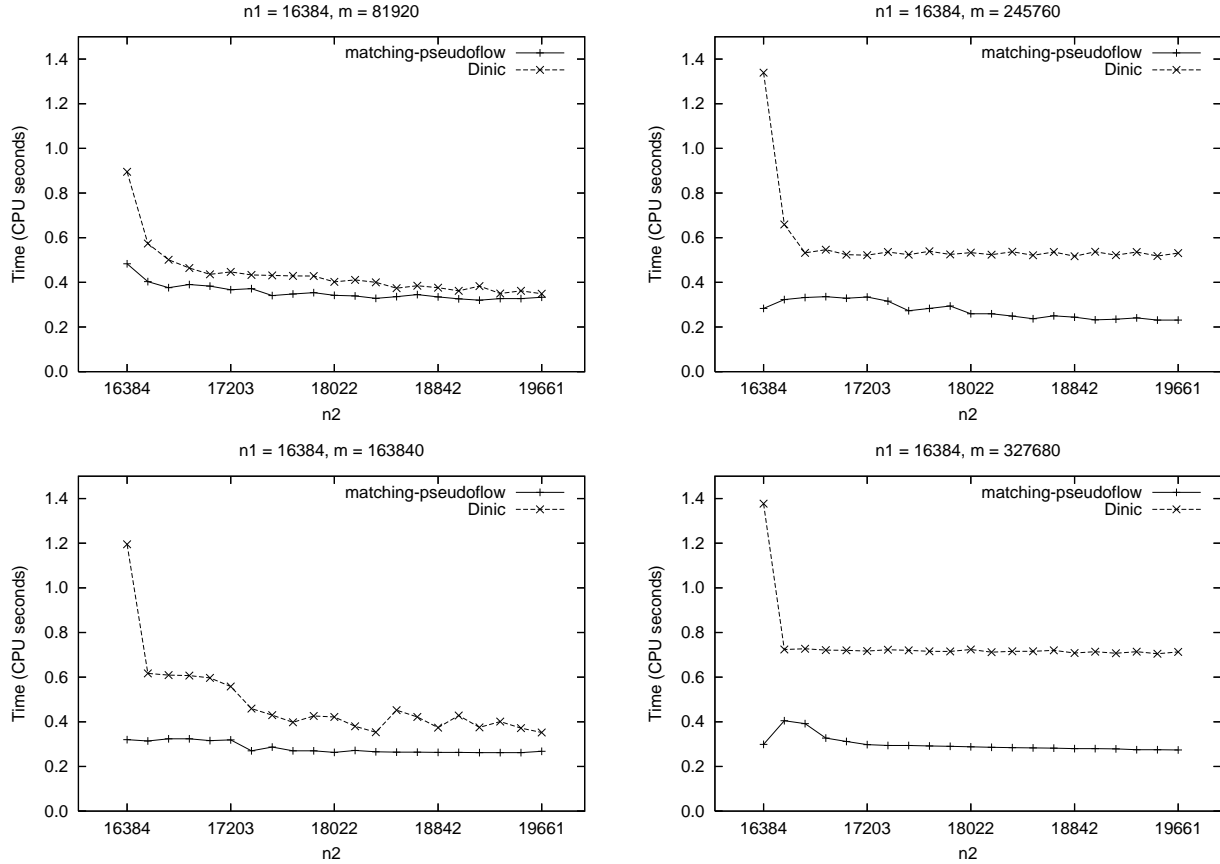


Figure 9: Run-times of the matching-pseudoflow and Dinic's algorithms on random unbalanced instances.

There are two key observations to be made from the results.

1. The **matching-pseudoflow** algorithm is more robust to imbalances in the graph. For Dinic’s algorithm, the balanced instances are the hardest to solve; even small imbalances in the graph ($n_2 = 1.01n_1$) make drastic differences to the run-time.
2. The run-time of Dinic’s algorithm goes up with the number of edges, but the run-time of the **matching-pseudoflow** is virtually independent of the number of edges. In fact, the hardest instances for the **matching-pseudoflow** appear to be the ones with fewest arcs.

6.4 Test instances

We tested the algorithms on the seven problem families (**hilo**, **fewg**, **manyg**, **grid**, **hexa**, **rope**, and **zipf**) used by Cherkassky et al. [12]. All the benchmark instances were balanced, i.e., $n_1 = n_2$. The instances are described in greater detail in Section D of the appendix.

For each instance family, we report the results of our experiments for

- Five pseudoflow implementations: **pseudo_lo_lifo** and **pseudo_hi_wave**, which were found in initial testing to be the fastest variants for the lowest and highest label algorithms respectively, **pseudo_lo_free**, **pseudo_hi_free**, and **matching-pseudoflow**. All the pseudoflow variants were initialized with a greedy matching.
- Three implementations of Cherkassky et al. [12]: **pr_bim_hi**, **pr_bim_lo**, and the best implementation among **pr_bim_fifo**, **bim_dfs**, **bim_bfs**, **bim_ar**, and **bim_lds**. The **pr_bim_hi** and **pr_bim_lo** implementation were tested on all families to compare them to the free-arcs pseudoflow variants.
- Implementations **abmp** and **dinic**.

6.5 Results

- **Hi-lo:** The run-times and operation counts for **hilo** instances are presented in Figure 10 and Table 3 respectively.

The **hilo** family was designed to be much harder for the highest label push-relabel algorithm than the lowest label variant. As expected, **pseudo_hi_free** and **bim_hi_free** are the slowest, though the former is faster than the latter. The **pseudo_lo_free** variant is the fastest of all algorithms, and is more than twice as fast as **bim_lo_free**.

Interestingly, the **pseudo_hi_wave** is faster than **pseudo_lo_lifo**, showing once again that pseudoflow and push-relabel have very different behavior.

The **pseudo_hi_wave** and **bim_bfs** algorithms show the best scaling behavior and are likely to be faster than **pseudo_lo_free** on larger instances. The **matching-pseudoflow** algorithm shows poor scaling behavior; it is faster than **dinic** on smaller instances but becomes slower on large instances.

The **pseudo_hi_wave** algorithm performs fewer arc scans and pushes (the dominant operations) than **pseudo_lo_free**, yet is slower. This suggests that the simplicity of the free-arcs implementations result in performance gains due to simplicity of code (which often leads to better compiler optimization).

- **Fewg:** The run-times and operation counts for **fewg** instances are presented in Figure 11 and Table 4 respectively.

The **pseudo_hi_free** and **pseudo_lo_free** algorithms are the fastest, and are more than twice as fast as the next-best algorithms (**pr_bim_hi** and **pr_bim_lo**). The difference seems to be in the number of arc scans performed.

The **matching-pseudoflow** and **dinic** algorithms are the slowest, though **matching-pseudoflow** is faster on all instance sizes.

- **Manyg:** The run-times and operation counts for **manyg** instances are presented in Figure 12 and Table 5 respectively.

The results are similar to the `fewg` instances. The `pseudo_hi_free` and `pseudo_lo_free` algorithms are the fastest, and are more than twice as fast as the next-best implementations (`pr_bim_hi` and `pr_bim_lo`), which is reflected in the number of arc scans performed.

While the `matching-pseudoflow` implementation is faster than `dinic` on all instance sizes, `dinic` appears to scale better and is likely to be faster on larger instances.

- **Grid:** The run-times and operation counts for `grid` instances are presented in Figure 13 and Table 6 respectively.

The scaling behavior of all the pseudoflow variants is extremely non-robust, making a comparison of the algorithms difficult. However, the `pseudo_hi_free` variant is the fastest on all instance sizes with the `pseudo_lo_free` variant close behind. These variants are more than twice as fast as the next-best algorithms (`pr_bim_hi` and `pr_bim_lo`).

The `matching-pseudoflow` algorithm is faster than `dinic`, although its scaling behavior is not robust.

The `pseudo_hi_free`, `pseudo_lo_free`, `pr_bim_hi` and `pr_bim_lo` algorithms did not perform any global re-labeling. Hence, this would be a good family to understand the fundamental differences between the four implementations. We see that the push-relabel variants perform a greater number of each of the operations; however, it is difficult to draw strong conclusions due to the non-robust scaling behavior of the pseudoflow variants.

- **Hexa:** The run-times and operation counts for `hexa` instances are presented in Figure 14 and Table 7 respectively.

The `pseudo_hi_free` and `pseudo_lo_free` algorithms are the fastest, followed by `pr_bim_lo` which is 1.5–1.8 times slower, which is reflected in the number of arc scans performed.

The `matching-pseudoflow` algorithm is faster than `dinic` by a similar factor.

- **Rope:** The run-times and operation counts for `rope` instances are presented in Figure 15 and Table 8 respectively.

This was the only family where `abmp` showed good performance, and is the fastest of all algorithms. The `matching-pseudoflow` is only marginally slower. Both `matching-pseudoflow` and `pseudo_lo_free` scale better than `abmp` and are likely to be faster on larger instances. The `matching-pseudoflow` algorithm is much faster than `dinic`, while the `pseudo_hi_free` algorithm is an order of magnitude faster than `pr_bim_hi`.

The operation counts do not provide much insight.

- **Zipf:** The run-times and operation counts for `zipf` instances are presented in Figure 16 and Table 9 respectively.

The `pseudo_hi_free` algorithm is the fastest, with `matching-pseudoflow` close behind. The next best algorithm is `pseudo_lo_free` (note that this is the only family in which `pseudo_lo_free` is not the best or nearly best algorithm).

The difference between the highest and lowest label variants seems to be due to the fact that no global relabels are triggered in the highest label variant, while the lowest label variants perform one relabel.

7 Discussion

We developed several variants of the pseudoflow algorithm for bipartite matching. One variant, the `matching-pseudoflow` algorithm was shown to have the best-known theoretical complexity for the problem. While the `matching-pseudoflow` could be viewed as a specialized implementation of Dinic’s algorithm, we believe that the `matching-pseudoflow` is a natural extension of the generic pseudoflow algorithm, whereas Dinic’s algorithm requires a greater degree of adaptation from its widely-accepted form. We also compared the `matching-pseudoflow` to Dinic’s algorithm to point out the key differences between the two algorithms.

We also developed several implementations of our algorithms and compared them to the fastest available codes based on the push-relabel algorithm. We draw the following conclusions from our experiments.

- Our best implementation was faster than that of Cherkassky et al. [12] on each problem family tested. The `psuedo_lo_free` algorithm was the fastest or nearly fastest algorithm in six of the seven instance classes tested. On the remaining family (`zipf`), it was the third-fastest implementation and was within a factor of 2 of the fastest implementation. We hence declare this to be the best pseudoflow variant overall and recommend that it be the algorithm of choice when solving bipartite matching problems.
- The `pseudo_lo_free` variant was generally faster than the `pseudo_hi_free` variant. This is consistent with the behavior of push-relabel where the lowest label variant was found to be faster than the highest label variant. However, in the regular pseudoflow variant (without free arcs), the highest label variant was generally faster than the lowest label variant.
- While the `psuedo_lo_free` and `pseudo_hi_free` could be viewed as special implementations of the push-relabel algorithm with a two-edge push, they are uniformly faster than the push-relabel implementations of Cherkassky et al. [12].

This difference is not due only to the different global relabeling frequency. In the `grid` instances where no global relabeling was performed, push-relabel variants performed more operations such as arc scans and pushes than the pseudoflow variants.

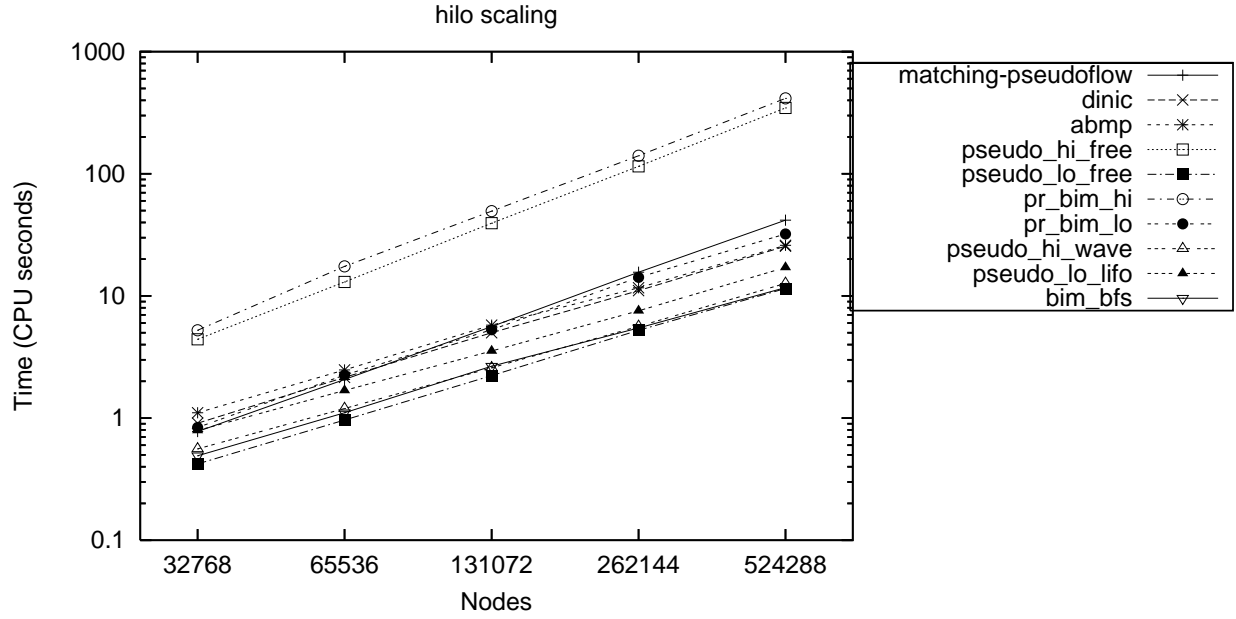
- Although implementations based on the regular pseudoflow algorithm (i.e., without free arcs) were faster than push-relabel for unit capacity networks [10], their performance is unimpressive for bipartite matching. This is surprising given that bipartite matching is a special case of unit capacity networks. In general, `pseudo_hi_wave` and `pseudo_lo_lifo` were at least a factor of 2 slower than the fastest algorithm. However, their performance was comparable to that of `pr_bim_hi` and `pr_bim_lo` on four of the families.
- The `matching-pseudoflow` algorithm is generally faster than `dinic`, and is nearly the fastest algorithm on two instance families. This is particularly interesting because the `matching-pseudoflow` algorithm could be viewed as an efficient implementation of Dinic’s algorithm. Past experimental studies [25, 26, 12] have dismissed Dinic’s algorithm as not being competitive in practice. However, the results here show that a careful implementation of Dinic’s algorithm (i.e., the `matching-pseudoflow`) can be very efficient in practice.
- The experiments comparing `matching-pseudoflow` and `dinic` on random graphs clearly shows that the theoretically efficient global relabeling procedure is efficient in practice as well.

On benchmark instances, `matching-pseudoflow` often performed a much greater number of global relabels. This is because `matching-pseudoflow` generates the layered network only until the lowest labeled layer of excess nodes and finds a blocking flow in this network, while `dinic` creates a layered network consisting of *all* excess nodes in the network and finds a blocking flow in this network.

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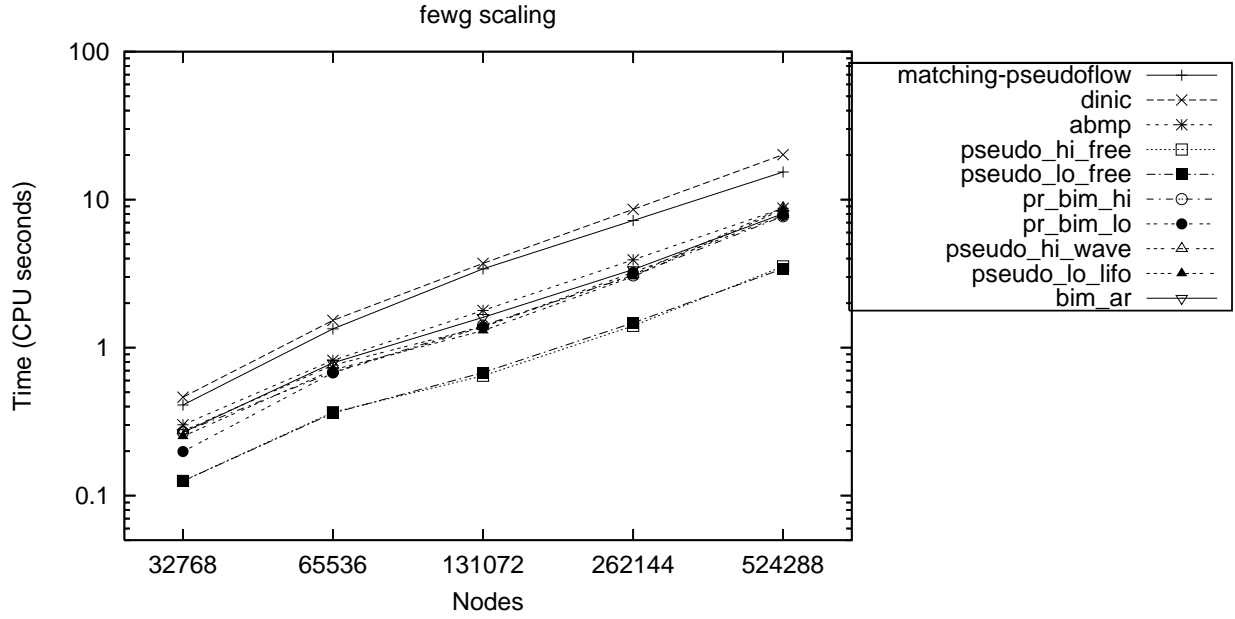


n, m	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 163,795	0.559 (1.325)	0.791 (1.877)	4.413 (10.468)	0.422 (1.000)	0.776 (1.841)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
65,536 327,635	1.106 (2.623)	0.906 (2.150)	5.238 (12.424)	0.838 (1.987)	0.492 (1.166)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
131,072 655,315	1.198 (1.241)	1.685 (1.746)	13.020 (13.486)	0.965 (1.000)	2.071 (2.146)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
262,144 1,310,675	2.482 (2.571)	2.154 (2.231)	17.431 (18.056)	2.250 (2.330)	1.106 (1.145)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
524,288 2,621,395	2.580 (1.158)	3.546 (1.592)	39.460 (17.714)	2.228 (1.000)	5.613 (2.520)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
524,288 2,621,395	5.774 (2.592)	5.001 (2.245)	49.463 (22.205)	5.279 (2.370)	2.658 (1.193)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
524,288 2,621,395	5.636 (1.077)	7.546 (1.442)	114.889 (21.957)	5.232 (1.000)	15.646 (2.990)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
524,288 2,621,395	11.640 (2.225)	11.040 (2.110)	140.445 (26.841)	14.175 (2.709)	5.462 (1.044)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
524,288 2,621,395	12.677 (1.102)	17.139 (1.490)	346.412 (30.123)	11.500 (1.000)	41.756 (3.631)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
524,288 2,621,395	25.991 (2.260)	25.574 (2.224)	414.229 (36.020)	32.184 (2.799)	11.695 (1.017)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow

Figure 10: Actual and relative run times for hilo instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 163,795	Arc scans	1,056,128	1,646,324	8,192,128	762,861	—
	Pushes	148,981	217,719	1,667,353	174,689	—
	Relabels	237,005	380,975	804,735	61,909	—
	Updates	—	—	116	4	36
	Mergers	71,035	70,219	841,869	95,537	—
	Depth	2.1	3.1	—	—	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	2,371,060	—	26,636,291	3,996,898	1,654,465
	Pushes	—	—	3,257,726	378,564	37,491
	Relabels	—	—	1,094,037	142,325	—
	Updates	8	9	33	4	—
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	2,236,728	3,327,699	22,912,731	1,701,580	—
	Pushes	306,091	432,966	4,640,923	386,262	—
65,536 327,635	Relabels	504,631	770,496	2,262,820	141,875	—
	Updates	—	—	164	4	48
	Mergers	143,112	140,653	2,336,846	209,515	—
	Depth	2.1	3.1	—	—	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	5,020,503	—	75,561,817	10,206,026	3,424,066
	Pushes	—	—	9,285,463	928,912	75,863
	Relabels	—	—	3,111,263	363,578	—
	Updates	8	9	47	6	—
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	4,561,083	6,732,126	64,839,782	3,997,936	—
	Pushes	634,653	894,858	13,079,076	891,346	—
	Relabels	1,030,516	1,559,513	6,426,451	343,113	—
	Updates	—	—	235	5	69
131,072 655,315	Mergers	287,326	282,516	6,572,306	478,441	—
	Depth	2.2	3.2	—	—	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	10,920,981	—	212,891,250	23,772,193	7,020,559
	Pushes	—	—	26,111,312	2,115,535	153,071
	Relabels	—	—	8,766,397	845,714	—
	Updates	9	11	67	6	—
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	9,169,155	13,506,106	180,439,780	8,529,365	—
	Pushes	1,355,311	1,816,976	36,312,845	1,890,103	—
	Relabels	2,071,512	3,128,909	17,927,990	739,821	—
	Updates	—	—	330	5	95
	Mergers	577,092	566,032	18,221,958	1,010,587	—
	Depth	2.3	3.2	—	—	—
262,144 1,310,675		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	23,615,817	—	598,255,607	53,222,206	14,603,340
	Pushes	—	—	73,733,362	4,695,511	307,005
	Relabels	—	—	24,701,383	1,899,744	—
	Updates	10	11	94	7	—
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	19,549,566	27,470,958	521,211,735	21,068,218	—
	Pushes	2,816,525	3,897,696	104,676,332	4,590,985	—
	Relabels	4,442,913	6,371,506	51,874,420	1,882,722	—
	Updates	—	—	477	5	131
	Mergers	1,162,414	1,133,943	52,469,238	2,426,565	—
	Depth	2.4	3.4	—	—	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	48,060,944	—	1,694,565,825	122,850,193	29,173,255
524,288 2,621,395	Pushes	—	—	208,586,161	10,589,122	619,162
	Relabels	—	—	69,908,716	4,377,687	—
	Updates	10	11	133	8	—

Table 3: Operation counts for hilo instances.

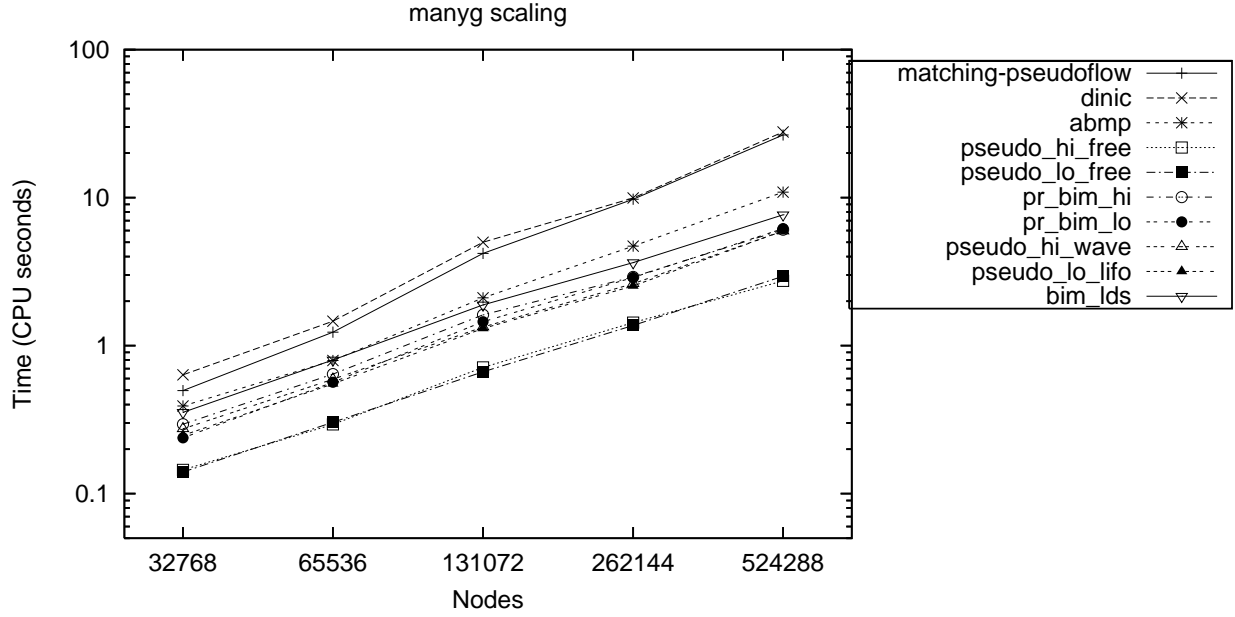


n, m					
32,768 82,095	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.2736 (2.178)	0.2516 (2.003)	0.1258 (1.002)	0.1256 (1.000)	0.4104 (3.268)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	0.3012 (2.398)	0.4628 (3.685)	0.268 (2.134)	0.1988 (1.583)	0.2674 (2.129)
65,536 163,656	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.7614 (2.106)	0.7072 (1.956)	0.3664 (1.013)	0.3616 (1.000)	1.3392 (3.704)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	0.8208 (2.270)	1.529 (4.228)	0.6816 (1.885)	0.6774 (1.873)	0.7886 (2.181)
131,072 327,600	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	1.383 (2.144)	1.3032 (2.020)	0.6452 (1.000)	0.6794 (1.053)	3.4068 (5.280)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	1.7844 (2.766)	3.7112 (5.752)	1.413 (2.190)	1.381 (2.140)	1.6012 (2.482)
262,144 654,952	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	3.1136 (2.219)	3.0152 (2.149)	1.403 (1.000)	1.473 (1.050)	7.2344 (5.156)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	3.9232 (2.796)	8.5908 (6.123)	3.0812 (2.196)	3.2264 (2.300)	3.3662 (2.399)
524,288 1,310,193	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	8.778 (2.566)	8.5982 (2.514)	3.5388 (1.035)	3.4206 (1.000)	15.3866 (4.498)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	8.7438 (2.556)	20.1562 (5.893)	7.7148 (2.255)	7.8182 (2.286)	8.0362 (2.349)

Figure 11: Actual and relative run times for fewg instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 82,095						
	Arc scans	181,799	172,026	204,682	196,601	—
	Pushes	161,912	170,298	88,909	85,202	—
	Relabels	85,772	80,903	35,429	34,384	—
	Updates	—	—	1	1	13
	Mergers	34,719	33,472	52,593	50,739	—
	Depth	4.7	5.1	1.7	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	419,220	—	618,981	472,235	764,076
	Pushes	—	—	120,143	92,889	20,213
	Relabels	—	—	41,827	31,899	—
	Updates	3	7	1	1	—
65,536 163,656						
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	329,471	309,255	396,201	397,374	—
	Pushes	319,904	326,880	173,575	172,192	—
	Relabels	149,212	138,432	68,828	69,677	—
	Updates	—	—	0	0	17
	Mergers	69,501	67,124	103,060	102,369	—
	Depth	4.6	4.9	1.7	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	932,152	—	893,740	921,534	1,367,342
	Pushes	—	—	188,754	185,257	40,512
	Relabels	—	—	64,203	63,540	—
	Updates	3	9	0	0	—
131,072 327,600						
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	793,163	778,705	1,000,363	965,803	—
	Pushes	740,609	824,823	431,014	414,934	—
	Relabels	390,966	385,902	179,456	174,645	—
	Updates	—	—	1	1	18
	Mergers	140,555	135,400	248,063	240,022	—
	Depth	5.3	6.1	1.7	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	2,308,131	—	3,093,455	2,727,033	3,083,830
	Pushes	—	—	577,100	503,949	81,001
	Relabels	—	—	203,909	177,560	—
	Updates	4	9	1	1	—
262,144 654,952						
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,569,306	1,558,767	2,001,868	1,941,552	—
	Pushes	1,673,973	1,816,616	860,458	832,131	—
	Relabels	768,873	771,057	358,377	350,394	—
	Updates	—	—	1	1	19
	Mergers	279,554	270,026	495,325	481,161	—
	Depth	6.0	6.7	1.7	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	4,587,565	—	5,984,012	5,971,654	6,322,937
	Pushes	—	—	1,132,298	1,123,517	162,055
	Relabels	—	—	399,311	398,476	—
	Updates	4	9	1	1	—
524,288 1,310,193						
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	3,288,620	3,260,495	4,063,834	4,086,478	—
	Pushes	4,550,107	4,342,098	1,762,001	1,746,771	—
	Relabels	1,639,415	1,639,381	737,480	741,989	—
	Updates	—	—	1	1	19
	Mergers	559,478	539,665	1,011,196	1,003,581	—
	Depth	8.1	8.0	1.7	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	9,040,958	—	12,902,902	12,909,703	12,968,662
	Pushes	—	—	2,404,536	2,393,865	323,979
	Relabels	—	—	851,428	852,664	—
	Updates	4	9	1	1	—

Table 4: Operation counts for fewg instances.

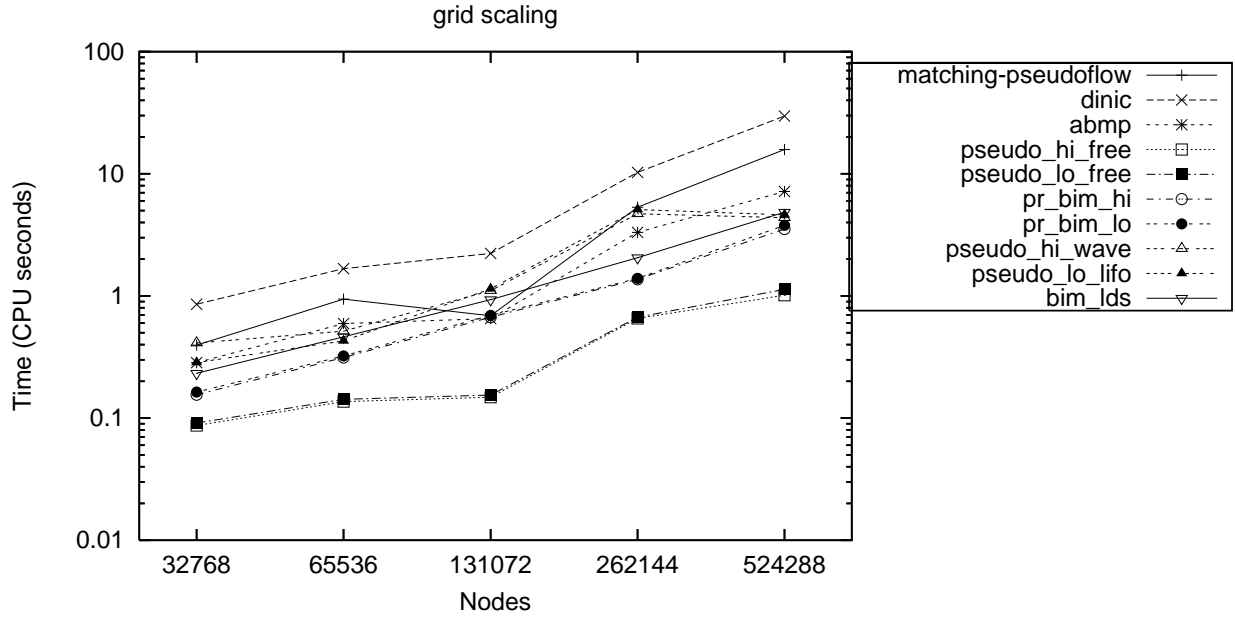


n, m					
32,768 82,077	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.2742 (1.956)	0.2474 (1.765)	0.1448 (1.033)	0.1402 (1.000)	0.497 (3.545)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	0.3914 (2.792)	0.6356 (4.534)	0.2948 (2.103)	0.2386 (1.702)	0.354 (2.525)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.5922 (2.018)	0.5512 (1.879)	0.2934 (1.000)	0.3036 (1.035)	1.2324 (4.200)
65,536 163,719	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	0.794 (2.706)	1.4626 (4.985)	0.6418 (2.187)	0.566 (1.929)	0.801 (2.730)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	1.3346 (2.008)	1.3086 (1.968)	0.713 (1.073)	0.6648 (1.000)	4.1976 (6.314)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	2.105 (3.166)	4.9986 (7.519)	1.6166 (2.432)	1.4522 (2.184)	1.884 (2.834)
262,144 654,959	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	2.62 (1.912)	2.53 (1.846)	1.4326 (1.046)	1.3702 (1.000)	9.7666 (7.128)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	4.6992 (3.430)	9.955 (7.265)	2.9092 (2.123)	2.8892 (2.109)	3.6358 (2.653)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	5.9466 (2.169)	5.9696 (2.177)	2.7418 (1.000)	2.9456 (1.074)	26.5288 (9.676)
524,288 1,310,160	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	10.8922 (3.973)	27.7888 (10.135)	6.0768 (2.216)	6.1736 (2.252)	7.6666 (2.796)

Figure 12: Actual and relative run times for manyg instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 82,077	Arc scans	192,549	184,781	248,068	232,279	–
	Pushes	132,243	152,523	106,419	99,685	–
	Relabels	91,643	88,978	44,199	41,694	–
	Updates	–	–	1	1	21
	Mergers	35,779	33,840	61,348	57,981	–
	Depth	3.7	4.5	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	652,624	–	692,732	550,212	839,879
	Pushes	–	–	131,103	104,488	45,333
	Relabels	–	–	46,067	36,361	–
	Updates	4	9	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	327,169	331,021	404,774	399,765	–
	Pushes	284,422	306,709	177,838	173,486	–
65,536 163,719	Relabels	147,392	152,630	71,146	70,413	–
	Updates	–	–	0	0	23
	Mergers	69,667	67,297	105,191	103,015	–
	Depth	4.1	4.6	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	1,224,438	–	1,002,185	928,942	1,371,693
	Pushes	–	–	202,282	186,948	85,372
	Relabels	–	–	69,369	64,205	–
	Updates	4	9	1	0	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	931,930	920,992	1,130,186	1,112,806	–
	Pushes	671,855	779,392	482,875	473,092	–
	Relabels	481,255	481,382	205,411	203,835	–
	Updates	–	–	1	1	29
131,072 327,587	Mergers	143,409	136,491	273,993	269,102	–
	Depth	4.7	5.7	1.8	1.8	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	3,148,755	–	3,840,908	3,296,800	4,337,322
	Pushes	–	–	694,463	604,050	200,422
	Relabels	–	–	249,093	216,005	–
	Updates	5	14	2	2	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,670,193	1,667,586	2,095,030	2,100,661	–
	Pushes	1,449,080	1,594,757	903,041	896,141	–
	Relabels	833,720	843,721	380,154	382,628	–
	Updates	–	–	1	1	31
	Mergers	282,170	270,932	516,616	513,166	–
	Depth	5.1	5.9	1.7	1.7	–
262,144 654,959		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	6,435,332	–	6,718,883	6,584,312	7,801,467
	Pushes	–	–	1,238,835	1,205,123	388,275
	Relabels	–	–	440,603	430,411	–
	Updates	5	15	2	2	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	2,905,666	2,920,705	3,979,979	3,900,105	–
	Pushes	2,815,922	3,351,759	1,721,001	1,668,690	–
	Relabels	1,380,457	1,412,220	717,924	703,407	–
	Updates	–	–	1	1	29
	Mergers	562,195	540,975	990,694	964,539	–
	Depth	5.0	6.2	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	11,189,842	–	10,377,914	10,398,863	13,327,851
524,288 1,310,160	Pushes	–	–	2,001,388	1,993,899	738,286
	Relabels	–	–	699,959	702,447	–
	Updates	4	13	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow

Table 5: Operation counts for manyg instances.

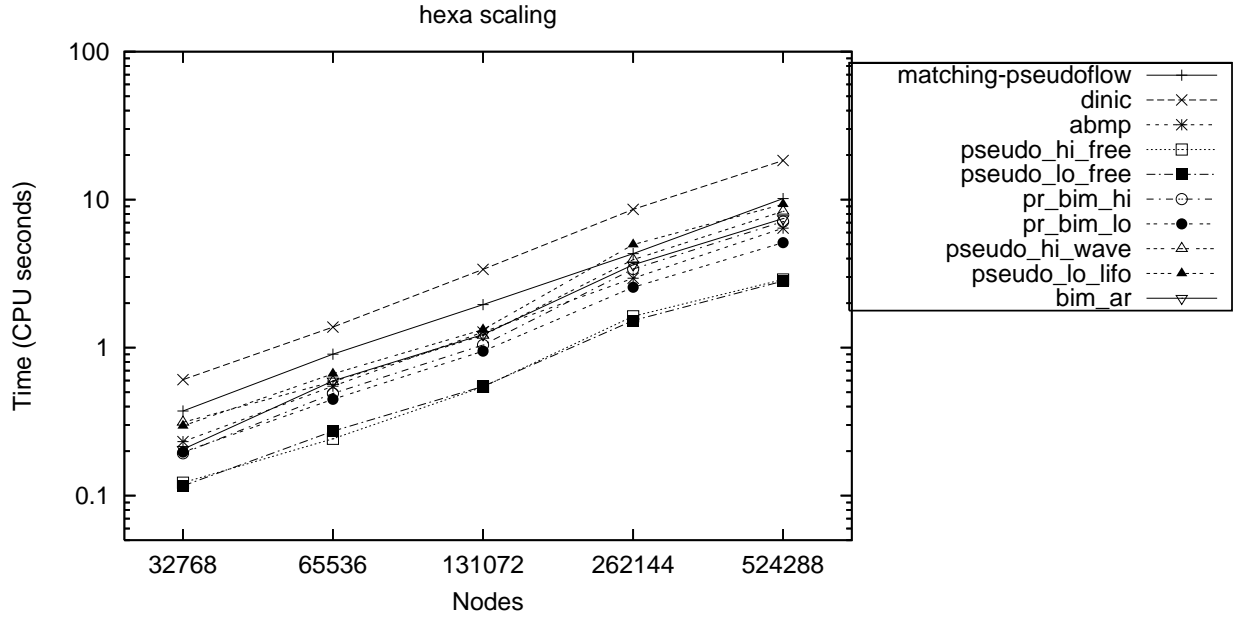


n, m					
32,768 98,304	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.412 (4.736)	0.286 (3.283)	0.087 (1.000)	0.091 (1.046)	0.396 (4.552)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	0.284 (3.269)	0.856 (9.839)	0.155 (1.784)	0.163 (1.878)	0.233 (2.676)
65,536 196,608	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.514 (3.764)	0.428 (3.132)	0.137 (1.000)	0.143 (1.044)	0.941 (6.889)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	0.595 (4.354)	1.670 (12.227)	0.313 (2.288)	0.322 (2.359)	0.462 (3.382)
131,072 393,216	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	1.106 (7.464)	1.142 (7.707)	0.148 (1.000)	0.154 (1.040)	0.687 (4.636)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	0.653 (4.404)	2.227 (15.030)	0.670 (4.524)	0.693 (4.673)	0.932 (6.291)
262,144 786,432	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	4.702 (7.184)	5.093 (7.780)	0.655 (1.000)	0.670 (1.023)	5.315 (8.120)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	3.319 (5.070)	10.269 (15.687)	1.369 (2.091)	1.392 (2.126)	2.057 (3.143)
524,288 1,572,864	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	4.365 (4.304)	4.612 (4.548)	1.014 (1.000)	1.133 (1.117)	15.789 (15.568)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	7.158 (7.058)	29.642 (29.227)	3.535 (3.486)	3.787 (3.734)	4.855 (4.787)

Figure 13: Actual and relative run times for grid instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 98,304	Arc scans	132,434	131,001	154,177	153,498	–
	Pushes	221,811	172,938	58,471	57,805	–
	Relabels	38,819	38,902	20,801	20,689	–
	Updates	–	–	0	0	10
	Mergers	32,806	32,075	37,427	37,094	–
	Depth	6.8	5.4	1.6	1.6	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	334,958	–	284,015	279,646	463,747
	Pushes	–	–	56,947	56,028	30,079
	Relabels	–	–	17,714	17,358	–
	Updates	1	6	0	0	–
65,536 196,608		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	250,158	249,170	219,660	216,645	–
	Pushes	347,171	293,260	87,502	86,312	–
	Relabels	70,092	70,482	27,238	26,745	–
	Updates	–	–	0	0	11
	Mergers	64,404	63,225	60,135	59,540	–
	Depth	5.4	4.6	1.5	1.4	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	714,586	–	513,351	509,895	871,303
	Pushes	–	–	101,531	100,682	57,070
	Relabels	–	–	31,510	31,205	–
	Updates	1	4	0	0	–
131,072 393,216		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	500,303	492,658	182,216	182,216	–
	Pushes	807,174	773,513	89,741	89,741	–
	Relabels	154,580	151,290	12,103	12,103	–
	Updates	–	–	0	0	10
	Mergers	124,654	121,033	77,639	77,639	–
	Depth	6.5	6.4	1.2	1.2	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	591,966	–	1,105,938	1,083,652	1,672,692
	Pushes	–	–	216,402	211,837	107,777
	Relabels	–	–	65,785	63,924	–
	Updates	0	4	0	0	–
262,144 786,432		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,006,773	999,686	980,473	936,799	–
	Pushes	2,263,080	1,543,859	390,886	377,017	–
	Relabels	282,641	282,317	129,561	122,832	–
	Updates	–	–	0	0	15
	Mergers	257,599	253,230	260,979	254,045	–
	Depth	8.8	6.1	1.5	1.5	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	3,724,682	–	2,023,942	2,012,307	3,375,505
	Pushes	–	–	399,144	396,445	221,951
	Relabels	–	–	124,220	123,233	–
	Updates	2	6	0	0	–
524,288 1,572,864		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	2,183,335	2,162,769	1,267,945	1,266,556	–
	Pushes	1,769,132	1,827,967	513,985	512,656	–
	Relabels	747,854	751,059	123,635	123,640	–
	Updates	–	–	0	0	27
	Mergers	528,271	503,973	388,065	387,400	–
	Depth	3.3	3.6	1.3	1.3	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_lds
	Arc scans	7,139,976	–	5,644,084	5,604,995	8,585,263
	Pushes	–	–	1,175,331	1,152,474	509,287
	Relabels	–	–	360,887	359,436	–
	Updates	2	10	0	0	–

Table 6: Operation counts for grid instances.

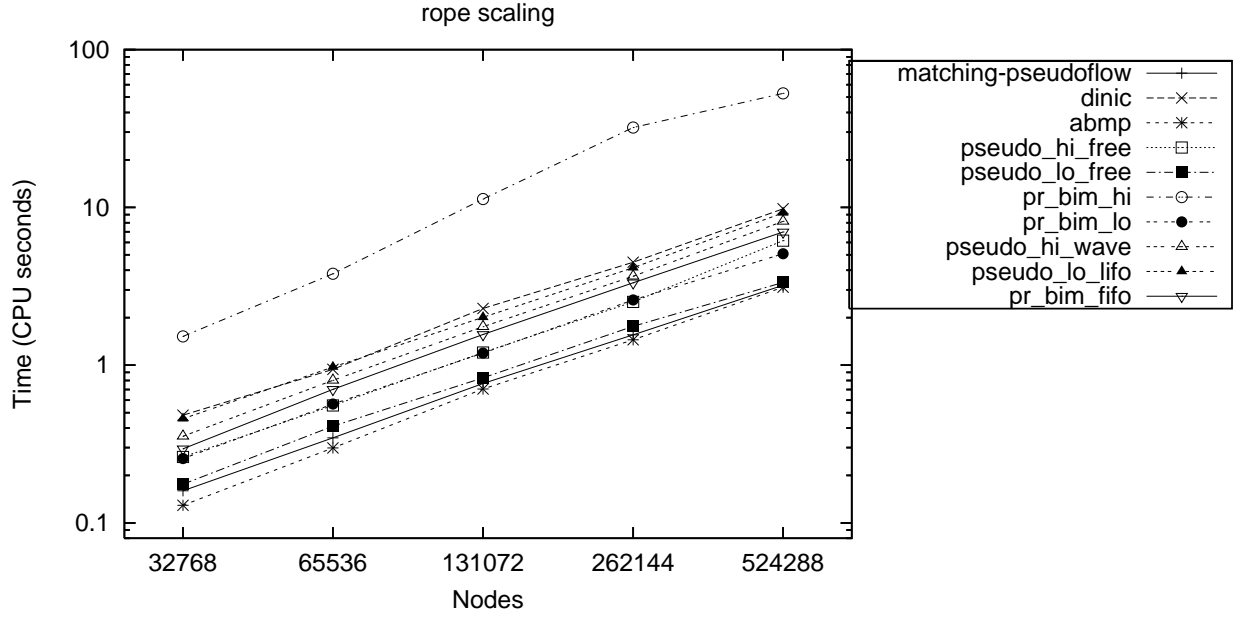


n, m					
32,768 98,304	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.313 (2.686)	0.297 (2.544)	0.122 (1.050)	0.117 (1.000)	0.373 (3.199)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	0.232 (1.986)	0.608 (5.218)	0.193 (1.657)	0.197 (1.691)	0.206 (1.767)
65,536 196,608	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.589 (2.434)	0.666 (2.750)	0.242 (1.000)	0.273 (1.127)	0.904 (3.736)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	0.549 (2.269)	1.379 (5.698)	0.490 (2.024)	0.448 (1.850)	0.596 (2.463)
131,072 393,216	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	1.210 (2.228)	1.325 (2.441)	0.543 (1.000)	0.547 (1.007)	1.955 (3.600)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	1.275 (2.348)	3.382 (6.229)	1.042 (1.919)	0.947 (1.744)	1.228 (2.261)
262,144 786,432	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	3.932 (2.579)	4.976 (3.264)	1.629 (1.068)	1.525 (1.000)	4.321 (2.834)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	2.946 (1.932)	8.602 (5.642)	3.397 (2.228)	2.556 (1.677)	3.633 (2.383)
524,288 1,572,864	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	8.344 (2.959)	9.318 (3.304)	2.890 (1.025)	2.820 (1.000)	10.176 (3.608)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	6.429 (2.280)	18.367 (6.512)	7.100 (2.517)	5.121 (1.816)	7.456 (2.643)

Figure 14: Actual and relative run times for hexa instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 98,304	Arc scans	248,522	218,196	225,952	218,890	–
	Pushes	170,434	245,992	79,222	77,716	–
	Relabels	92,615	79,810	30,985	31,041	–
	Updates	–	–	1	1	8
	Mergers	33,431	31,771	47,412	46,659	–
	Depth	5.1	7.7	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	369,751	–	535,122	470,222	789,152
	Pushes	–	–	92,042	82,743	18,931
	Relabels	–	–	32,278	28,995	–
	Updates	2	4	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	458,751	401,241	439,941	413,710	–
	Pushes	322,442	490,279	155,760	147,014	–
65,536 196,608	Relabels	165,117	140,819	60,672	57,802	–
	Updates	–	–	1	1	9
	Mergers	66,349	63,366	93,468	89,095	–
	Depth	4.9	7.7	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	717,931	–	876,664	907,438	1,248,027
	Pushes	–	–	153,849	153,994	37,924
	Relabels	–	–	52,928	53,413	–
	Updates	2	5	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	924,762	836,356	894,516	846,603	–
	Pushes	654,422	1,302,141	319,055	301,013	–
	Relabels	334,360	297,826	125,278	119,074	–
	Updates	–	–	1	1	10
131,072 393,216	Mergers	131,790	126,412	190,691	181,670	–
	Depth	5.0	10.3	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	1,724,821	–	2,215,824	1,837,612	3,025,906
	Pushes	–	–	374,792	304,199	75,885
	Relabels	–	–	131,342	105,363	–
	Updates	2	5	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,678,375	1,464,907	1,705,404	1,726,744	–
	Pushes	1,424,488	1,958,491	609,616	611,618	–
	Relabels	581,244	492,883	236,370	242,745	–
	Updates	–	–	1	0	11
	Mergers	264,336	254,686	367,131	368,132	–
	Depth	5.4	7.7	1.7	1.7	–
262,144 786,432		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	3,204,871	–	3,721,038	3,403,393	5,406,653
	Pushes	–	–	655,858	585,720	152,163
	Relabels	–	–	226,987	201,954	–
	Updates	2	5	1	0	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	3,891,630	3,719,454	4,006,828	3,931,486	–
	Pushes	3,938,423	5,722,941	1,419,725	1,385,077	–
	Relabels	1,428,979	1,365,452	573,130	566,263	–
	Updates	–	–	1	1	13
	Mergers	533,359	514,992	834,474	817,150	–
	Depth	7.4	11.1	1.7	1.7	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_ar
	Arc scans	7,494,811	–	11,608,019	8,224,669	13,204,537
524,288 1,572,864	Pushes	–	–	1,856,977	1,305,988	304,642
	Relabels	–	–	662,983	455,882	–
	Updates	2	6	1	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow

Table 7: Operation counts for hexa instances.

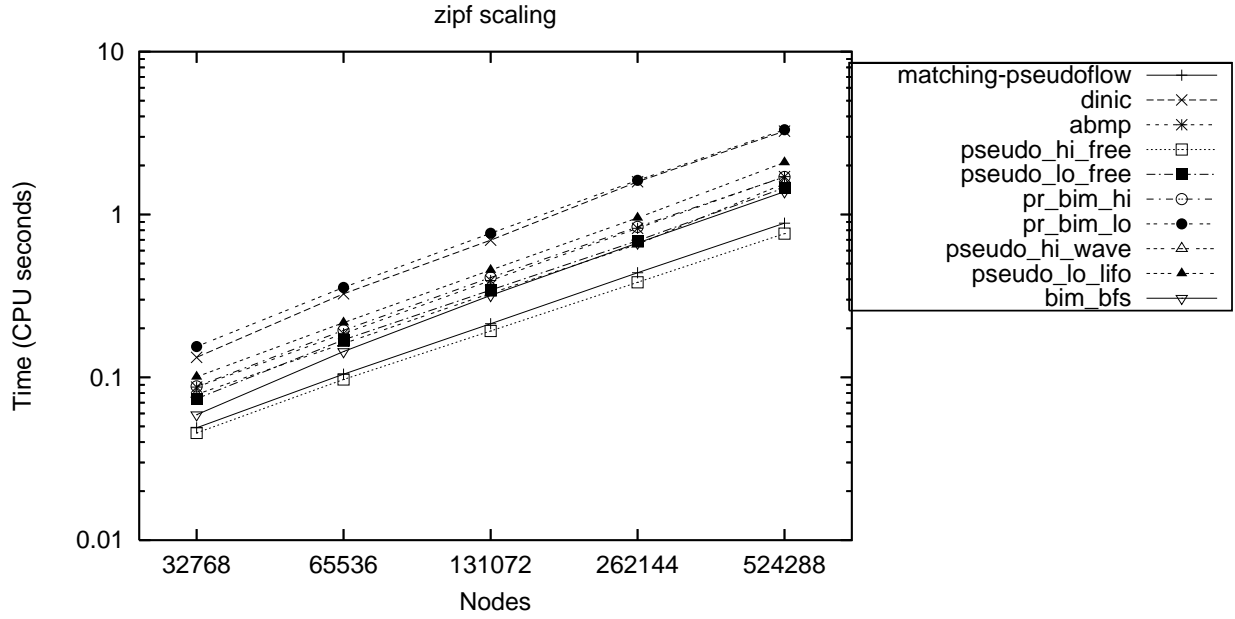


n, m					
32,768 98,304	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.354 (2.743)	0.457 (3.540)	0.262 (2.026)	0.176 (1.361)	0.160 (1.237)
	abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
65,536 196,608	0.129 (1.000)	0.482 (3.729)	1.521 (11.769)	0.256 (1.981)	0.295 (2.280)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.801 (2.685)	0.973 (3.261)	0.555 (1.860)	0.411 (1.376)	0.346 (1.159)
131,072 393,216	abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	0.298 (1.000)	0.942 (3.157)	3.797 (12.725)	0.567 (1.900)	0.703 (2.357)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
262,144 786,432	1.753 (2.485)	2.009 (2.847)	1.201 (1.703)	0.830 (1.177)	0.765 (1.084)
	abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	0.706 (1.000)	2.289 (3.244)	11.307 (16.024)	1.194 (1.692)	1.567 (2.221)
524,288 1,572,864	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	3.667 (2.528)	4.141 (2.855)	2.515 (1.734)	1.762 (1.215)	1.557 (1.073)
	abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
524,288 1,572,864	1.450 (1.000)	4.490 (3.095)	32.118 (22.144)	2.593 (1.788)	3.329 (2.295)
	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	8.152 (2.614)	9.243 (2.964)	6.154 (1.973)	3.343 (1.072)	3.210 (1.029)
524,288 1,572,864	abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	3.119 (1.000)	9.804 (3.143)	52.828 (16.938)	5.088 (1.631)	6.979 (2.238)

Figure 15: Actual and relative run times for rope instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 98,304	Arc scans	450,472	640,668	284,307	185,589	—
	Pushes	109,632	177,015	125,871	67,952	—
	Relabels	184,689	266,637	33,881	16,449	—
	Updates	—	—	4	1	12
	Mergers	50,793	59,201	71,126	42,166	—
	Depth	2.2	3.0	1.8	1.6	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	Arc scans	234,915	—	4,807,442	698,951	706,610
	Pushes	—	—	948,187	111,356	121,627
	Relabels	—	—	265,278	34,320	34,764
	Updates	1	5	8	1	1
65,536 196,608		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	928,042	1,286,990	627,108	430,849	—
	Pushes	216,963	355,423	278,442	157,025	—
	Relabels	383,272	535,995	78,260	42,779	—
	Updates	—	—	5	2	13
	Mergers	101,391	118,642	155,604	94,896	—
	Depth	2.1	3.0	1.8	1.7	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	Arc scans	447,591	—	10,954,608	1,399,024	1,401,884
	Pushes	—	—	2,189,028	223,019	239,673
	Relabels	—	—	604,427	68,655	68,626
	Updates	1	6	9	1	1
131,072 393,216		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,948,058	2,601,575	1,377,293	788,584	—
	Pushes	445,041	707,218	658,013	288,835	—
	Relabels	812,234	1,085,864	172,964	71,711	—
	Updates	—	—	6	1	15
	Mergers	203,135	237,263	361,772	177,184	—
	Depth	2.2	3.0	1.8	1.6	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	Arc scans	910,755	—	27,236,410	2,800,225	2,820,512
	Pushes	—	—	5,320,568	446,756	484,653
	Relabels	—	—	1,493,655	137,355	138,212
	Updates	1	6	11	1	1
262,144 786,432		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	3,961,794	5,202,682	2,553,059	1,498,842	—
	Pushes	878,458	1,417,061	1,250,749	544,346	—
	Relabels	1,657,356	2,171,661	322,220	135,953	—
	Updates	—	—	5	1	17
	Mergers	406,396	474,495	690,909	337,708	—
	Depth	2.2	3.0	1.8	1.6	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	Arc scans	1,787,955	—	61,856,682	5,608,445	5,617,592
	Pushes	—	—	12,354,203	895,346	962,001
	Relabels	—	—	3,412,444	275,013	274,651
	Updates	1	6	13	1	1
524,288 1,572,864		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	8,149,084	10,433,210	6,288,325	3,043,754	—
	Pushes	1,790,739	2,880,818	3,345,473	1,118,770	—
	Relabels	3,424,575	4,354,052	814,531	273,814	—
	Updates	—	—	7	1	19
	Mergers	815,731	951,915	1,803,806	690,455	—
	Depth	2.2	3.0	1.9	1.6	—
		abmp	dinic	pr_bim_hi	pr_bim_lo	pr_bim_fifo
	Arc scans	3,876,650	—	166,032,545	11,209,692	11,242,609
	Pushes	—	—	33,096,689	1,788,801	1,926,834
	Relabels	—	—	9,134,295	549,426	549,353
	Updates	1	6	17	1	1

Table 8: Operation counts for rope instances.



n, m					
32,768 98,304	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.079 (1.724)	0.101 (2.211)	0.046 (1.000)	0.074 (1.618)	0.049 (1.075)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	0.087 (1.912)	0.133 (2.908)	0.088 (1.930)	0.154 (3.386)	0.059 (1.294)
65,536 196,608	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.161 (1.664)	0.217 (2.233)	0.097 (1.000)	0.170 (1.755)	0.105 (1.078)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	0.186 (1.913)	0.326 (3.363)	0.194 (2.004)	0.356 (3.674)	0.144 (1.487)
131,072 393,216	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.328 (1.703)	0.456 (2.369)	0.193 (1.000)	0.344 (1.785)	0.213 (1.108)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	0.394 (2.044)	0.696 (3.616)	0.411 (2.132)	0.768 (3.985)	0.319 (1.656)
262,144 786,432	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	0.657 (1.713)	0.954 (2.489)	0.383 (1.000)	0.691 (1.802)	0.439 (1.146)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	0.823 (2.148)	1.589 (4.145)	0.840 (2.190)	1.622 (4.230)	0.668 (1.742)
524,288 1,572,864	pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	1.550 (2.034)	2.090 (2.742)	0.762 (1.000)	1.458 (1.913)	0.885 (1.161)
	abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	1.709 (2.242)	3.247 (4.261)	1.702 (2.233)	3.316 (4.351)	1.380 (1.811)

Figure 16: Actual and relative run times for zipf instances.

n, m		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
32,768 98,304	Arc scans	72,958	72,086	93,511	72,960	–
	Pushes	21,603	20,103	31,891	22,717	–
	Relabels	48,253	48,278	23,393	17,549	–
	Updates	–	–	0	1	5
	Mergers	19,124	17,866	23,446	18,859	–
	Depth	1.1	1.1	1.4	1.2	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	117,350	–	194,865	286,754	143,561
	Pushes	–	–	55,897	57,937	12,250
	Relabels	–	–	25,683	32,771	–
	Updates	1	2	0	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	142,930	141,199	194,193	142,776	–
	Pushes	41,179	38,480	63,280	43,503	–
65,536 196,608	Relabels	95,105	94,902	48,395	34,562	–
	Updates	–	–	0	1	6
	Mergers	36,814	34,501	46,292	36,404	–
	Depth	1.1	1.1	1.4	1.2	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	246,974	–	380,736	564,877	282,227
	Pushes	–	–	109,182	113,575	23,733
	Relabels	–	–	51,179	65,539	–
	Updates	1	2	0	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	276,323	280,119	410,131	278,417	–
	Pushes	78,569	73,818	122,384	83,038	–
	Relabels	182,747	196,278	104,001	67,823	–
	Updates	–	–	0	1	6
131,072 393,216	Mergers	70,913	66,712	89,809	70,135	–
	Depth	1.1	1.1	1.4	1.2	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	473,993	–	753,217	1,117,726	555,996
	Pushes	–	–	216,199	223,694	45,932
	Relabels	–	–	103,400	131,075	–
	Updates	1	2	0	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	540,623	542,210	846,919	542,842	–
	Pushes	149,664	141,379	234,925	158,296	–
	Relabels	360,906	375,809	215,777	133,381	–
	Updates	–	–	0	1	6
	Mergers	136,426	128,788	173,329	135,015	–
	Depth	1.1	1.1	1.4	1.2	–
262,144 786,432		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	950,009	–	1,477,907	2,199,752	1,090,311
	Pushes	–	–	426,648	439,407	89,149
	Relabels	–	–	208,224	262,147	–
	Updates	1	2	0	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow
	Arc scans	1,066,914	1,052,889	1,610,434	1,106,222	–
	Pushes	286,868	271,869	436,616	306,668	–
	Relabels	726,440	720,922	417,641	275,043	–
	Updates	–	–	0	1	7
	Mergers	263,206	249,177	327,303	262,329	–
	Depth	1.1	1.1	1.3	1.2	–
		abmp	dinic	pr_bim_hi	pr_bim_lo	bim_bfs
	Arc scans	1,858,122	–	2,888,953	4,367,508	2,164,681
524,288 1,572,864	Pushes	–	–	830,304	873,298	173,285
	Relabels	–	–	412,878	524,291	–
	Updates	1	2	0	1	–
		pseudo_hi_wave	pseudo_lo_lifo	pseudo_hi_free	pseudo_lo_free	matching-pseudoflow

Table 9: Operation counts for zipf instances.

A Proof of Lemma 5.5

Proof: Our analysis of the number of stages is essentially the same as that of Dinic's algorithm as per Even and Tarjan [15] and Hopcroft and Karp [21].

By construction, each ℓ layered network guarantees at least one successful path, as some WT_1 node is reachable through a sequence of mergers of length ℓ . We divide the stages into two parts: the first part includes stages of labels no larger than $\sqrt{\kappa}$, and the second part consists of the stages with labels greater than $\sqrt{\kappa}$. Since the label of the lowest labeled strong node strictly increases in each stage, the number of stages in the first part is at most $\sqrt{\kappa}$. We show that the second part can also have at most $\sqrt{\kappa}$ stages.

In the second part of the algorithm, the successful paths of length $L > \sqrt{\kappa}$ are equivalent to flow augmentations along a path of length $2L + 1$. We now observe that each WT_2 branch contains a residual arc of capacity 1 from root to child, and the set of residual arcs in the WT_2 branches in any layer $p > 0$ of the network forms a valid cut in the residual graph. This is since it separates the roots in this layer and nodes with label greater than p from the children in the layer and nodes with label less than p as in Figure 17.

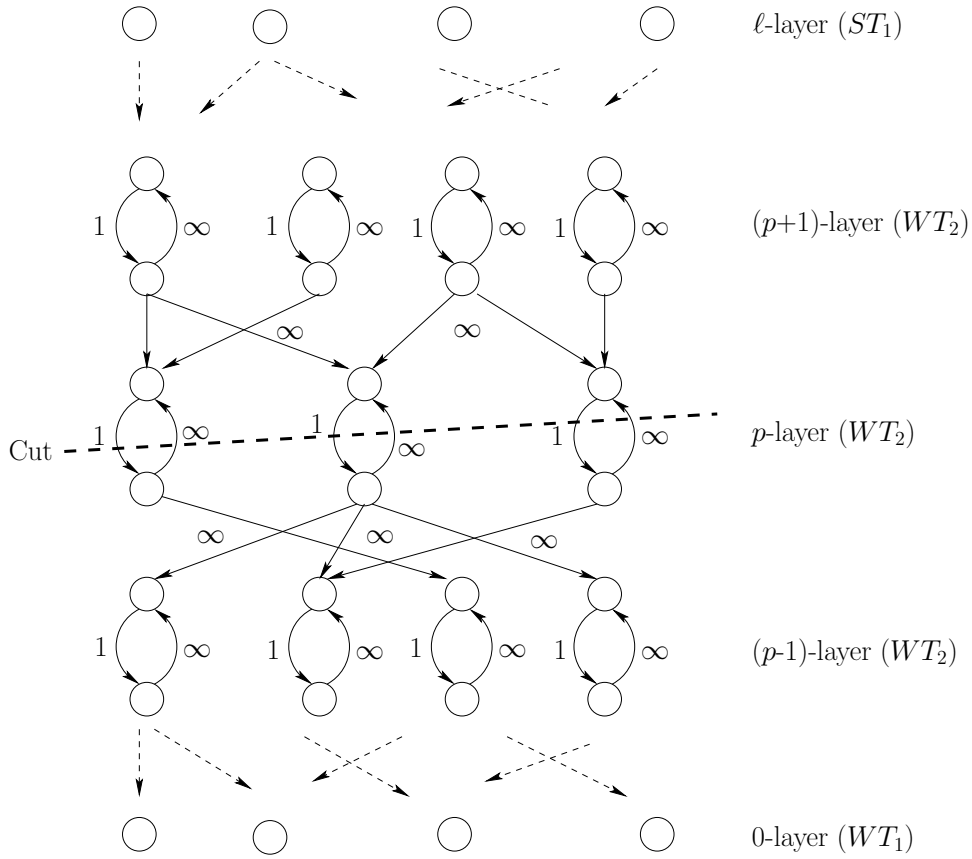


Figure 17: Arcs in the branches of a layer form a cut in the residual graph.

Thus the maximum flow value in the residual graph at the beginning of part two is no larger than the smallest number of branches in one of the layers. Since the layered network consists of at most κ branches and the number of layers is L , then the maximum flow in the residual graph can be no larger than $\frac{\kappa}{L}$, which in the second part is no larger than $\sqrt{\kappa}$. Thus the total number of augmentations in part two is at most $\sqrt{\kappa}$. Since each layered network guarantees at least once augmentation, there are at most $\sqrt{\kappa}$ stages in the second part of the algorithm. ■

B Complexity using word operations

We show here how to use boolean operations to improve the complexity of the **matching-pseudoflow** algorithm. The characteristic vector of out-neighbors of each V_1 -node v is maintained as a binary word $\text{OUT}(v)$ of length n_2 . $\text{OUT}(v)$ is a word where the i^{th} bit is 1 if there is an arc from $v \in V_1$ to $i \in V_2$. We also maintain a characteristic vector of in-neighbors list as a word $\text{IN}(v)$ of length n_1 for each node in $v \in V_2$. $\text{IN}(v)$ is a word where the i^{th} bit is 1 if the arc from $i \in V_1$ to $v \in V_2$ exists.

These words are maintained *in addition to the adjacency list* which is a linked list of in and out neighbors for each node in the graph. The words and the adjacency list are used in parallel to achieve the better time complexity of the approach using only words and that using only the adjacency list. When we say that the two are used in parallel we imply that the adjacency list and word operations are accessed and used alternately.

Using λ -bit word operations ($\lambda < n_1$), we break $\text{OUT}()$ and $\text{IN}()$ into a concatenation of λ -bit words, and perform operations on these words. Each of these λ -bit words is called a λ -word and the j^{th} λ -word is denoted by $\text{OUT}^j(v)$.

Three boolean operations are used:

1. **LEAD**: Given a word W , $\text{lead}(W)$ returns the index of the leading non-zero bit in W , and 0 if all bits are 0.
2. **AND**: Given two words A and B of the same length, $A \wedge B$ is a word whose i^{th} bit is 1 iff the i^{th} bits of A and B are 1, and 0 otherwise.
3. **OR**: Given two words A and B of the same length, $A \vee B$ is a word whose i^{th} bit is 1 if the i^{th} bit of A or B (or both) is 1, and 0 otherwise.

If we wish to perform any of the above operations on a word of k bits using word operations on words of λ bits where $\lambda < k$, each k -bit word operation can be done in $O(\frac{k}{\lambda})$ steps.

Any boolean operation (\wedge , \vee , **lead**) on a λ -word counts as a single operation. Given two nodes $i \in V_1$ and $j \in V_2$, the bits corresponding to the arc (i, j) in $\text{IN}(j)$ and $\text{OUT}(i)$ can be accessed and modified in $O(1)$.

Initialization For each node $v \in V_1$, we look at the next arc in its out neighbors in the adjacency list. If this arc does not lead to an unmatched WT_1 node, we perform a $\text{lead}(\text{OUT}^1(v))$ operation. If $\text{lead}(\text{OUT}^1(v))$ equals 0, we return to the adjacency list and look at the next arc. Again, if this arc does not lead to an unmatched V_2 -node, a **lead** operation is performed on the next unscanned λ -word ($\text{OUT}^2(v)$). This procedure of looking at the next λ -word and the next arc in the adjacency list until an unmatched V_2 neighbor is found, or the end of the list is reached. In the adjacency list, either a neighboring WT_1 branch is found or all the neighbors are exhausted in at most κ arc scans for each V_1 -node. Thus, there are at most $n_1\kappa$ arc scans. Further, each arc is looked at most once, so the complexity is $O(\min\{n_1\kappa, m\})$.

In $\text{OUT}(v)$, either a neighboring WT_1 branch is found or all the neighbors are exhausted in $O(n_2/\lambda)$ operations. A neighboring WT_1 branch, if it exists, is thus found in $O(\min\{n_1\kappa, m, \frac{n_1n_2}{\lambda}\})$.

Once a merger is executed, the bits corresponding to the merger arc in the $\text{IN}()$ and $\text{OUT}()$ words must be changed. Since there are at most κ mergers during initialization, and each requires $O(1)$ work, the work done to maintain these words is $O(\kappa)$.

Claim B.1 *The work done in initialization using λ -words is $O(\min\{n_1\kappa, m, \frac{n_1n_2}{\lambda}\})$.*

Building the 1-layer Similar to the initialization, for each child v of a WT_2 branch, we search for a merger arc by looking in parallel at the next neighbor in the adjacency list and performing a **lead**() operation on the next λ -word $\text{OUT}(v)$. The search terminates either when a WT_1 neighbor is found or the end of the list is reached, which occurs in $\min\{\kappa, n_2/\lambda\}$ operations. Since there are at most κ nodes that are children of a WT_2 branch and each arc is scanned at most once in the entire algorithm, the total work to generate the 1-layer of the layered network throughout the algorithm is $O(\min\{\kappa^2, n_2\kappa/\lambda, m\})$.

Building the layered network We now describe the use of word operations in generating a layered network upwards from the 1-layer. With the exception of the ℓ -layer, all the branches in the layered network are WT_2 branches. We first discuss labeling the WT_2 branches, and later discuss how to find the ℓ -layer.

We construct words for the sub-graph induced only by the nodes in the WT_2 branches. That is, each node $v \in V_1$ which is the child of a WT_2 branch has an associated word $\text{SUB-OUT}(v)$ containing the subset of its out-neighbor nodes that are in WT_2 branches, the length of which is at most κ . Note that this is different from $\text{OUT}(v)$ which is a word of length n_2 and contains all out neighbors of v , not just those that are in WT_2 branches.

The i^{th} bit of $\text{SUB-OUT}(v)$ is 1 if an arc exists from v to the i^{th} node which is a root of a WT_2 branch. Similarly, the roots of the WT_2 branches have a word $\text{SUB-IN}(v)$ of size at most κ representing the in-neighbors of v that are children in a WT_2 branch. We will use $\text{SUB-IN}()$ to build the layered network and $\text{SUB-OUT}()$ while pushing flow through this network.

Initially, $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ are empty since there are no WT_2 branches. As WT_2 branches are created during the algorithm, $\text{SUB-IN}()$ $\text{SUB-OUT}()$ words are created for each of the nodes in these branches. At any point in the algorithm, $\text{SUB-OUT}(v)$ is a subset of $\text{OUT}(v)$ and $\text{SUB-IN}(v)$ is a subset of $\text{IN}(v)$ that contains only those bits that correspond to nodes that are in WT_2 branches. The relation between $\text{IN}()$, $\text{OUT}()$, $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ are shown in Figure 18. The matrix formed by the $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ words is referred to as the SUB-matrix.

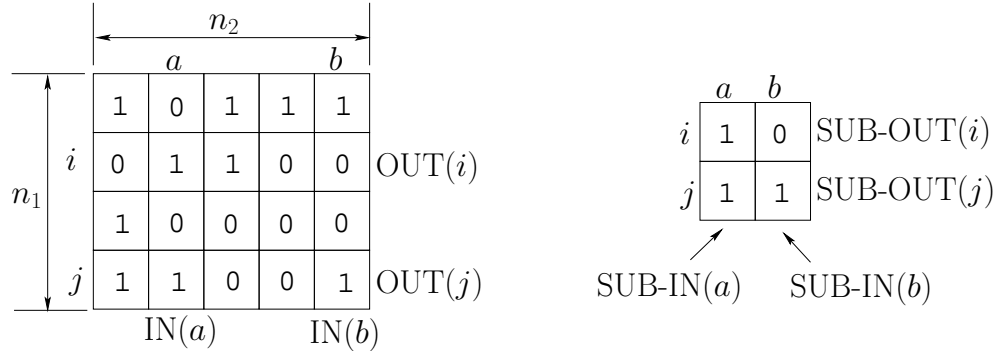


Figure 18: $\text{IN}()$, $\text{OUT}()$, $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ at some point during the algorithm when nodes $i, j \in V_1$, and $a, b \in V_2$ are in WT_2 branches.

Each time a singleton node (either WT_1 or ST_1 branch) becomes part of a WT_2 branch either during initialization or later in the algorithm, we add a bit corresponding to that node to the existing $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ words, and create a new word for that node. This is equivalent to adding a row and column to the SUB-matrix. The total work throughout the algorithm is $O(\kappa^2)$ since there are $O(\kappa)$ words each of size $O(\kappa)$, and adding a bit to the words is an $O(1)$ operation.

Two more types of words are needed to construct the layered network.

1. A $V_2\text{LAYER}(k)$ word (the characteristic vector of each layer) indicating the V_2 -nodes contained in each layer $1 \leq k \leq \kappa$. The length of the word is at most κ and a bit of $V_2\text{LAYER}(k)$ is 1 if a V_2 -node corresponding to that bit is in layer k . All the $V_2\text{LAYER}()$ words are set to 0 at the beginning of each stage.
2. A $V_1\text{LAYER}(k)$ word (the characteristic vector of each layer) indicating the V_1 -nodes contained in each layer $1 \leq k \leq \kappa$. The length of the word is at most κ and a bit of $V_1\text{LAYER}(k)$ is 1 if a V_1 -node corresponding to that bit is in layer k . All the $V_1\text{LAYER}()$ words are set to 0 at the beginning of each stage.
3. A REACHED word of size κ that keeps track of V_1 -nodes that have *not* been reached by the upward breadth-first-search in each stage to create the layered network. The i^{th} bit is 0 if that node has been assigned to a layer, and 1 otherwise. All the bits of this word are set to 1 at the beginning of each stage.

Once we have computed the 1-layer of the network, $V_1\text{LAYER}(1)$ is populated with 1 in the locations of the nodes in the 1-layer. REACHED is then populated with 0 in the locations of the V_1 -nodes in the 1-layer.

Step 1: Build $V_1\text{LAYER}(1)$

1	0	1	0	0	0	0
---	---	---	---	---	---	---

$a \qquad b$

Step 2: Initialize $\text{REACHED} = \neg (V_1\text{LAYER}(1))$

0	1	0	1	1	1	1
---	---	---	---	---	---	---

Step 3: Build $V_2\text{LAYER}(1)$

0	1	0	1	0	0	0
---	---	---	---	---	---	---

\nearrow Parent(a) \nwarrow Parent(b)

Step 4: Get $\text{SUB-IN}(\text{Parent}(a))$

1	0	0	1	0	0	0
---	---	---	---	---	---	---

Step 5: Get $\text{SUB-IN}(\text{Parent}(b))$

1	0	1	0	1	0	0
---	---	---	---	---	---	---

Step 6: Compute $V_1\text{LAYER}(2)$

0	0	0	1	1	0	0
---	---	---	---	---	---	---

$c \qquad d$

$$(\text{SUB-IN}(\text{Parent}(a)) \vee \text{SUB-IN}(\text{Parent}(b))) \wedge (\text{REACHED})$$

Figure 19: Finding V_1 -nodes in the 2-layer from $V_2\text{LAYER}(1)$ using word operations.

The $V_2\text{LAYER}(1)$ is now constructed by successively looking at each λ -word in $V_1\text{LAYER}(1)$ and performing a **lead()** operation on that word. If the result of the **lead()** operation is non-zero, then we know the index of a V_1 -node in the 1-layer, and its unique parent's bit is changed in the $V_2\text{LAYER}(1)$. The 1-bit corresponding to the output of the **lead()** operation is now set to 0, and another **lead()** operation is performed on the same word. Identifying a 1-bit and changing it continues until the result of the **lead()** operation is zero, in which case we move to the next λ -word and perform a **lead()** operation on that word to find a 1-bit.

Now, given V_2 -nodes $\{v_1, \dots, v_j\}$ in the 1-layer, $V_1\text{LAYER}(2)$ is

$$V_1\text{LAYER}(2) = (\text{SUB} - \text{IN}(v_1) \vee \text{SUB} - \text{IN}(v_2) \vee \dots \vee \text{SUB} - \text{IN}(v_j)) \wedge \text{REACHED}.$$

Figure 19 illustrates this procedure on an example.

The V_2 -nodes in the 2-layer are obtained from $V_1\text{LAYER}(2)$ (the parent of a V_1 -node in the 2-layer is a V_2 -node in the 2-layer). The **REACHED** word is updated, the $V_2\text{LAYER}(2)$ word is constructed a bit at a time using the V_2 -nodes in the 2-layer. As above, $V_1\text{LAYER}(3)$ is now constructed from the **SUB-IN()** words of V_2 -nodes in the 2-layer and **REACHED**. This continues until there are no more changes in **REACHED**.

The complexity of constructing the $V_2\text{LAYER}()$ from the $V_1\text{LAYER}()$ takes $O(\kappa^2/\lambda)$ throughout the stage. The number of word operations that result in finding a 1-bit in the $V_1\text{LAYER}$, and changing the corresponding bit in the $V_2\text{LAYER}$ is at most κ , since there are at most κ WT_2 branches. The number of word operations that result in not finding a 1-bit is $O(\kappa/\lambda)$ for each layer since each $V_1\text{LAYER}()$ is of length κ and we look at the next λ -word when we do not find a 1-bit. There are at most κ layers, so the work done in finding the $V_2\text{LAYER}()$ words given the $V_1\text{LAYER}()$ words is $O(\kappa^2/\lambda)$ per stage.

An operation is performed on each λ -word of **SUB-IN()** at most once for each node in a stage, and **SUB-IN()** is of length at most κ ; so the work to generate the layered network is $O(\kappa^2/\lambda)$. The work to update the **REACHED** word is $O(\kappa)$ per stage.

At this point, we have the two-edge distances of all WT_1 branches. To find the set of ST_1 immediately reachable from this set, we use **IN()** (not **SUB-IN()** since we want to reach nodes outside the set of WT_2 branches) and the incoming arcs in the adjacency list, in parallel, for each node to check if a ST_1 branch is reachable from this node. Finding the ℓ -layer is done analogously to finding the 1-layer. For each node v that is the root of a WT_2 branch, an incoming arc in the adjacency list is scanned for a ST_1 neighbor. If no merger is found, a **lead()** operation is performed on λ bits of the $\text{IN}^1(v)$ to check for an ST_1 neighbor. If no merger is found, the next arc in the adjacency list is looked at. This procedure of looking at the next arc in the incoming arcs in the adjacency list and performing a **lead()** on the next λ -word of $\text{IN}(v)$ in parallel continues until a ST_1 neighbor is found, or all the neighbors are exhausted. Since $\text{IN}(v)$ is a word of length n_1 , the end of this word is reached in $O(n_1/\lambda)$ operations. The end of the adjacency list is reached in at most κ arc scans of the adjacency list. Further, each arc is looked at most once so the total work done throughout the algorithm in checking for ST_1 neighbors is $O(\min\{\kappa n_1/\lambda, \kappa^2, m\})$.

We also maintain a word VISITED (of length at most κ) that keeps track of the branches that have been visited at each stage, i.e., the i^{th} bit of this word is 1 if the root of the branch has not been visited in that stage. All bits in this word are initially set to 1.

To push flow through the network, we use $V_2\text{LAYER}()$, VISITED , and $\text{SUB-OUT}()$ to identify a merger arc. For a node $v \in V_1$ of label p , the set of arcs from node v to an unvisited node of layer $p - 1$ is found by $V_2\text{LAYER}(p - 1) \wedge \text{SUB-OUT}(v) \wedge \text{VISITED}$. A $\text{lead}()$ operation on this resultant word gives a merger arc if it exists.

Each time a merger is found, one more branch becomes visited. Therefore, there can be at most κ mergers in each stage. Hence, there are at most κ word operations that lead to mergers, which takes $O(\kappa)$ work. Each time a node is revisited in a stage, the search for mergers starts from the last λ -word checked for a merger; so the work done in word operations that do not find a mergers is $O(\kappa/\lambda)$ per node per stage, which in $O(\kappa^2/\lambda)$ total work per stage. Updating VISITED requires $O(\kappa)$ work throughout the stage. Hence, work to push flow by executing mergers is $O(\kappa^2/\lambda)$ per stage.

Since each stage can have at most κ successful mergers and there are $O(\sqrt{\kappa})$ stages, the number of successful mergers is $O(\kappa^{3/2})$. The $\text{IN}()$, $\text{OUT}()$, $\text{SUB-IN}()$ and $\text{SUB-OUT}()$ words need to be updated each time a successful merger occurs. Each update takes $O(1)$, so the total work updating these words is $O(\kappa^{3/2})$.

Table 10 summarizes our complexity results for our algorithms with word operations.

Operation	Per stage	Total
Initialization	-	$O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\})$
Constructing 1-layer	-	$O(\min\{\kappa^2, \frac{n_2\kappa}{\lambda}, m\})$
Constructing ℓ -layer	-	$O(\min\{\kappa^2, \frac{n_1\kappa}{\lambda}, m\})$
Layered network - layers $2, \dots, \ell - 1$	$O(\kappa^2/\lambda)$	$O(\kappa^{2.5}/\lambda)$
Executing mergers	$O(\kappa^2/\lambda)$	$O(\kappa^{2.5}/\lambda)$
Creating SUB-IN and SUB-OUT	-	$O(\kappa^2)$
Updating SUB-IN , SUB-OUT , IN , and OUT	$O(\kappa)$	$O(\kappa^{3/2})$
TOTAL		$O(\min\{n_1\kappa, \frac{n_1n_2}{\lambda}, m\} + \kappa^2 + \kappa^{2.5}/\lambda)$

Table 10: Complexity summary of algorithm for bipartite matching with word operations.

C An alternative approach

We now show that it is possible to achieve the theoretical complexity of the **matching-pseudoflow** algorithm by a clever analysis of Hopcroft and Karp's matching algorithm [21]¹. Given graph $G = (V_1 \cup V_2, E)$, let the cardinality of the greedy matching be κ_g . Denote the nodes in the maximal matching by $V_g \subseteq V$, then $|V_g| = 2\kappa_g$.

Lemma C.1 $\kappa \leq 2\kappa_g$.

Proof: Every edge in the graph has at least one end point in V_g (otherwise, an edge with neither end point in V_g can be added to the matching, which contradicts maximality). Therefore, every edge in an optimal matching must also have at least one end point in V_g . Thus, the cardinality of the maximum matching is bounded by the cardinality of the set V_g , which is $2\kappa_g$. ■

For each $v \in V_g$, let $E_g(v)$ denote the set of edges that have one end point in v and the other end point in another node in V_g . We now construct a graph $G^* = (V_1 \cup V_2, E^*)$, where $E^* \subseteq E$ contains the following edges:

- (i) For every node $v \in V_g$ with degree $\leq 2\kappa_g$ in G , E^* contains all edges adjacent to v .

¹We thank an anonymous referee for this analysis

- (ii) For every node $v \in V_g$ with degree $> 2\kappa_g$ in G , E^* contains all edges in $E_g(v)$ and an arbitrary subset of $2\kappa_g - |E_g(v)|$ edges adjacent to v that are not in $E_g(v)$. That is, a subset of $2\kappa_g$ edges adjacent to v that contain all the edges in $E_g(v)$.

Each node $v \in V_g$ in G^* has at most $2\kappa_g$ neighbors by construction. Since every edge is adjacent to some node in V_g and $|V_g| = 2\kappa_g$, the total number of edges in E^* is at most $4\kappa_g^2$. Since $\kappa_g \leq \kappa$, E^* has $O(\min\{m, \kappa^2\})$ edges.

Theorem C.1 *A maximum matching in G^* has cardinality κ .*

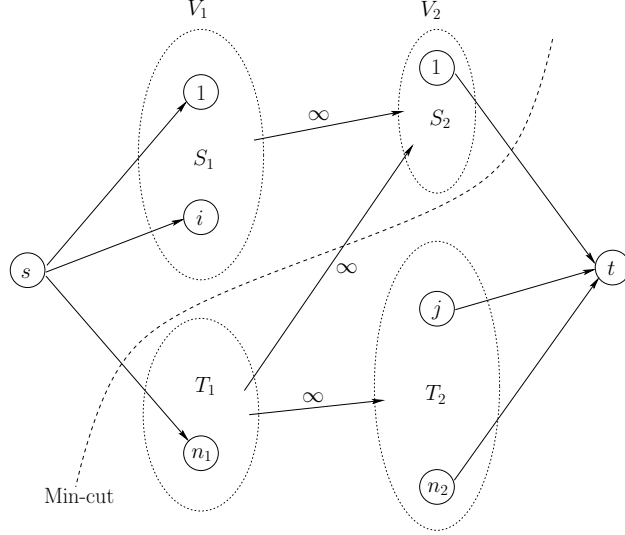


Figure 20: Minimum cut in G_{st}^* .

Proof: Let the cardinality of a maximum matching in G^* be denoted by κ^* . We obtain the maximum matching by solving for a minimum cut in a graph G_{st}^* obtained by adding a source node s , a sink node t , and unit capacity arcs from s to all nodes in V_1 and from all nodes in V_2 to t . Let the source set of the minimum cut in be $S^* = \{s\} \cup S_1 \cup S_2$ and the sink set be $T^* = \{t\} \cup T_1 \cup T_2$ where $S_1 \cup T_1 = V_1$ and $S_2 \cup T_2 = V_2$ as shown in Figure 20. Then, the capacity of this minimum cut is κ^* , i.e., $|T_1| + |S_2| = \kappa^*$.

The maximum matching in G is similarly obtained by solving for a minimum cut in a graph G_{st} (obtained by adding a source and a sink node, and arcs adjacent to the source and sink); this minimum cut has capacity $\kappa \leq n_1 \leq n_2$.

Suppose (for contradiction) that $\kappa^* < \kappa$. Then, S_1 , S_2 , T_1 , and T_2 are non-empty (if any of these sets were empty, then $\kappa^* = n_1$ which contradicts the assumption that $\kappa^* < \kappa \leq n_1$). The minimum s, t -cut (S^*, T^*) in G_{st}^* cannot be a finite cut in G_{st} since it has a capacity strictly less than the minimum cut in G_{st} . Then, there exists some arc (i, j) in G_{st} but not in G_{st}^* such that $i \in S_1$ and $j \in T_2$. Since the arc (i, j) was removed from G to generate G^* , it means that node i has exactly $2\kappa_g$ neighbors in G^* . Further, since i is in the source set of a finite cut in G_{st}^* , all the neighbors of i must belong to S_2 . That is, $|S_2| \geq 2\kappa_g$. We have shown that $\kappa^* > |S_2|$ since T_1 is non-empty. Therefore, $\kappa^* > |S_2| \geq 2\kappa_g \geq \kappa$, contradicting the assumption that $\kappa^* < \kappa$. \blacksquare

The above observations and theorem suggest the following algorithm:

1. Generate a maximal matching (takes $O(\min\{m, n_1\kappa\})$ work).
2. Construct graph G^* as described above (takes $O(\min\{m, \kappa^2\})$ work).
3. Solve for a maximum matching using the Hopcroft-Karp algorithm. Since the number of edges in G^* is $O(\min\{m, \kappa^2\})$, and the number of nodes is $O(\kappa)$, the complexity is $O(\sqrt{\kappa} \min\{m, \kappa^2\})$.

D Test instances

The descriptions of these instances is reproduced from Cherkassky et al. [12].

1. **Fewg and manyg**: These are random bipartite graphs where the vertices of each partition, V_1 and V_2 , are divided into k groups of equal size. For each vertex of the j -th group of V_1 the generator chooses y random neighbors from the $(i-1)$ -th through $(i+1)$ -th groups of V_2 (with wrap-around), where y is binomially distributed with mean d (thus $d = \text{mean vertex degree}$). The indices i and j are not related because vertices in V_1 are randomly shuffled before neighbors in V_2 are assigned. The two families we consider are **fewg**, where there are 32 groups, and **manyg**, where there are 256 groups; both have $d = 5$.

These classes were designed having in mind problems that can be reduced to bipartite matching, such as the maximum vertex-disjoint paths problem. In these problems the resulting graph in the reduction is bipartite, but if the original graph is planar or nearly planar each vertex will only have as neighbors vertices in the surrounding area.

2. **Hilo**: The hi-lo family of bipartite matching problems was designed to separate high and low vertex selection strategies for the push-relabel method. This generator creates a graph with a unique perfect matching and has been motivated by a generator of Kennedy [22].

Let $G = (V_1; V_2, E)$ be a graph produced by this generator. This graph is defined by three parameters, ℓ , k , and d . Vertices of V_1 are partitioned into ℓ groups, each containing k vertices. For $1 \leq i \leq k$, $1 \leq j \leq \ell$, we refer to the i -th vertex in group j by x_i^j . Vertices of V_2 are partitioned similarly, and y_i^j is defined similarly to x_i^j . Each vertex x_i^j is connected to vertices y_p^j for $\max(1, i-d) \leq p \leq i$ and, if $j < \ell$, to vertices y_p^{j+1} for $\max(1, i-d) \leq p \leq i$.

3. **Grid**: In class **grid**, each vertex $u \in V_1$ is connected to vertices $\{u+1, u-1, u+a, u-a, u+b, u-b, \dots\}$ where $\{1, a, b, \dots\}$ is a geometric progression. In our tests, we set the average degree of each node to 6.
4. **Hexa**: In class **hexa**, the vertices on each side are divided into n/b blocks of size b . One random bipartite hexagon is added between each block i on one side and each of the blocks $i+k$ on the other side, with $|k| \leq K$ for some K . The parameters b and K are chosen by the program in such a way that the average degree is correct (i.e., $3K/b = d$) but few pairs of hexagons have more than one vertex in common. In our tests, we set $d = 6$.
5. **Rope**: For the class **rope**, the vertices on each side are grouped into $t = n/d$ blocks of size d , numbered $V_1^0 \dots V_1^{t-1}$ and $V_2^0 \dots V_2^{t-1}$. Block i on one side is connected to block $i+1$ on the other side, for $i = 0, 1, \dots, t-2$; block V_1^{t-1} is connected to block V_2^{t-1} . Thus, the graph is a "rope" that is folded and twisted over itself, so that it zig-zags between the two sides, first up and then down. Consecutive pairs of blocks along the "rope" are connected alternately by perfect matchings ("m-type arcs") and random bipartite graphs of average degree $d-1$ ("r-type arcs"), beginning and ending with perfect matchings. The only maximum matching is a perfect one, consisting of all m -type arcs. In our tests, set $d = 6$.
6. **Zipf**: Each member of class **zipf** is a random bipartite graph where the arc between the i -th V_1 -node and the j -th V_2 -node has nominal probability roughly proportional to $1/(ij)$. Thus the graph is denser near the "core" vertices (those with small index), and thins out slowly towards the "periphery" (vertices with high index). In our experiments we set $d = 6$.